STERNGTH AND STIFFNESS OF FIBER-REINFORCED, PARTICULATE-MATRIX COMPOSITES

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ABSTRACT

The paper presents a comprehensive formulation of the stiffness and strength micromechanics for fiber-reinforced composites with the matrix reinforced with spherical particles. The solution is obtained in two steps: firstly, the stiffness (or strength) of a particulate matrix is evaluated; this data is subsequently employed to predict the properties of the fiber-reinforced material with the homogenized particulate matrix. As follows from numerical examples, the transverse strength and stiffness of the material can be significantly improved by adding a relatively small amount of stiff particles to the matrix. The effect of particles on the longitudinal strength and stiffness is less pronounced.

1. INTRODUCTION

The optimization of composite structures is usually concerned with either increasing their load-carrying capacity without additional weight or reducing weight without sacrificing the load-carrying capacity. In both situations it is necessary to increase the stiffness and strength of the structure. The straightforward approach to achieving enhanced properties is to use a stiffer high-strength material. An alternative approach employs spatially tailored structures with a variable over-the-surface stiffness. Functionally graded structures where the composition of the constituent phases varies through the thickness have also been considered [1]. The analysis in the present paper is concerned with enhancing the performance of composite structures by adding stiff inclusions (particles) to the fiber-reinforced material. While the benefits of this approach for the stiffness of composite materials have recently been demonstrated by Genin and Birman in [2], this paper did not address the issue of the effect of inclusions on the composite strength.

The present paper presents the formulation of the two-step micromechanical model for the stiffness and strength analysis of the material that consists of a particulate matrix (matrix with stiff spherical inclusions) that is reinforced by fibers. In particular, the strength of the material is characterized in two steps: first the strength of the particulate matrix and subsequently, the strength of the fiber-reinforced material with the homogenized isotropic matrix. As follows from numerical examples, particles embedded in the matrix can significantly increase both the stiffness and the strength of fiber-reinforced materials.

2. MICROMECHANICS OF FIBER-REINFORCED, PARTICULATE-MATRIX MATERIAL

Numerous methodologies of the micromechanical stiffness analysis of composites with arbitrary inclusions include the Mori-Tanaka model, the double-inclusion method, the models of Ponte Castaneda and Willis, the Kuster-Toksoz model, etc. The bounds for the tensor of stiffness have been suggested by Hashin and Shtrikman, Beran, Molyneux and McCoy, Gibiansky and Torquato, etc. The comprehensive review of these techniques is outside the scope of this paper, see for example, [3, 4].

Genin and Birman have recently extended the Benveniste solution based on the Mori-Tanaka method to characterize the tensor of stiffness of a fiber-reinforced, particulate-matrix material [2]. The stiffness of a cross-ply material evaluated using their approach was within the strict three-point bounds as long as the volume fraction of spherical inclusions and fibers remain relatively small. In the present paper the strength and stiffness analyses of a three-phase particulate-matrix fiber-reinforced material are conducted in two steps. We begin with the known micromechanical stiffness solution for a particulate matrix that is subsequently applied to evaluate the stiffness of a fiber-reinforced, particulate-matrix material. Then the strength of the matrix with inclusions (particles) is specified using the stiffness data. In turn, this solution enables us to predict the strength of the fiber-reinforced material with the homogeneous particulate matrix.

2.1. Two-step stiffness analysis

Consider a fiber-reinforced material where the matrix contains uniformly distributed and uniaxially aligned spheroidal inclusions. Throughout the paper it is assumed that the matrix is perfectly bonded to both fibers and spheroidal inclusions. The volume fraction of the inclusions within the matrix remains below 40%, so that the Mori-Tanaka approach is accurate [2]. Then the tensor of stiffness of the matrix with the particles can be obtained following Benveniste [5] in the form

$$\overline{\mathbf{L}}_{pm} = \overline{\mathbf{L}}_1 + f_2'\left(\overline{\mathbf{L}}_2 - \overline{\mathbf{L}}_1\right)\overline{\mathbf{T}}_2\left(f_1'\overline{\mathbf{I}} + f_2'\overline{\mathbf{T}}_2\right)^{-1} \tag{1}$$

where the subscripts “1” and “2” identify the matrix and particles, respectively, $\overline{\mathbf{L}}_i$ is a tensor of stiffness of the corresponding phase, $\overline{\mathbf{I}}$ is a fourth-order unit tensor, f_i' is the volume fraction of the i-th phase, and the prime indicates that these volume fractions are evaluated within the particulate matrix, i.e. $f_1' + f_2' = 1$. Furthermore,

$$\overline{\mathbf{T}}_2 = \left[\overline{\mathbf{I}} + \overline{\mathbf{S}}_2\overline{\mathbf{L}}_1^{-1}\left(\overline{\mathbf{L}}_2 - \overline{\mathbf{L}}_1\right)\right]^{-1} \tag{2}$$

is the coefficient tensor in the relation between the strain tensors in the matrix and in the particles

$$\overline{\varepsilon}_2 = \overline{\mathbf{T}}_2\overline{\varepsilon}_1 \tag{3}$$

The elements of the Eshelby tensor $\overline{\mathbf{S}}_2$ were obtained for spheroidal inclusions dependent on the aspect ratio (see, for example, [6]).

Once the tensor of stiffness of the particulate matrix has been obtained, it is possible to treat the matrix as a homogeneous medium that is also isotropic if the particles are spherical or if they are randomly oriented. Subsequently, we can apply a similar homogenization procedure to a unidirectional fiber-reinforced material considering fibers as aligned inclusions with an infinite aspect ratio. Accordingly, the tensor of stiffness of the fiber-reinforced, particulate-matrix material is

$$\overline{\mathbf{L}} = \overline{\mathbf{L}}_{pm} + f_3\left(\overline{\mathbf{L}}_3 - \overline{\mathbf{L}}_{pm}\right)\overline{\mathbf{T}}_3\left(f_{pm}\overline{\mathbf{I}} + f_3\overline{\mathbf{T}}_3\right)^{-1} \tag{4}$$

where f_3 and f_{pm} are the volume fractions of fibers and particulate matrix, respectively, $f_{pm} + f_3 = 1$, and

$$\overline{\mathbf{T}}_3 = \left[\overline{\mathbf{I}} + \overline{\mathbf{S}}_3\overline{\mathbf{L}}_{pm}^{-1}\left(\overline{\mathbf{L}}_3 - \overline{\mathbf{L}}_{pm}\right)\right]^{-1} \tag{5}$$

The Eshelby tensor for a fiber-reinforced material with an isotropic homogenous matrix, $\overline{\mathbf{S}}_3$, is given in [7]. Alternative micromechanical methods for a fiber-reinforced material include the well-known Halpin-Tsai or simplified mechanics of materials solutions.

2.2. Strength of the particulate matrix

The analysis of the particulate matrix is conducted by assumption that the particles are stiffer and have higher strength than the matrix. Accordingly, failure occurs within the matrix in the immediate vicinity to the particle-matrix interface. The pioneering work of Eshelby [8] presents the stresses just outside a spheroidal inclusion. This solution was further formalized for a particulate material in [6] and in [9] where the analytical results [6] were shown in good agreement with the finite element analysis.

Consider a particulate matrix subject to uniaxial tension $\hat{\sigma}_{11}$ (The in-plane coordinates referred to hereafter are denoted by 1 and 2). The failure condition can be determined using the maximum principal stress criterion in the matrix or by von Mises' criterion. In both cases, the strength can be normalized in terms of the tensile strength of the matrix s_{mT}.

The stresses in the matrix (in the 1-2 plane) just outside the inclusions are

$$\sigma_{11,m} = \hat{\sigma}_{11}\left[1 + \frac{(1 - f_2')(b_1 p_1 + 2b_2 p_2)}{(1+\nu_1)(1-2\nu_1)} + \frac{p_1\cos^2\theta + p_2(\nu_1 + \sin^2\theta)}{1-\nu_1^2}\cos^2\theta\right] = F_{11}(\theta)\hat{\sigma}_{11}$$

$$\sigma_{22,m} = \hat{\sigma}_{11}\left[\frac{(1 - f_2')(b_3 p_1 + (b_4 + b_5) p_2)}{(1+\nu_1)(1-2\nu_1)} + \frac{p_1\cos^2\theta + p_2(\nu_1 + \sin^2\theta)}{1-\nu_1^2}\sin^2\theta\right] = F_{22}(\theta)\hat{\sigma}$$

$$\sigma_{33,m} = \hat{\sigma}_{11}\left[\frac{(1-f_2')(b_3 p_1 + (b_4+b_5)p_2)}{(1+\nu_1)(1-2\nu_1)} + \frac{\nu_1 p_1 \cos^2\theta + p_2(1+\nu_1\sin^2\theta)}{1-\nu_1^2}\right] = F_{33}(\theta)\hat{\sigma}_{11}$$

$$\sigma_{12,m} = -\hat{\sigma}_{11}\frac{p_1\cos^2\theta + p_2(\nu_1+\sin^2\theta)}{1-\nu_1^2}\sin\theta\cos\theta = F_{12}(\theta)\hat{\sigma}_{11} \qquad \sigma_{13},_m = \sigma_{23},_m = 0 \tag{6}$$

where the coefficients b_j and p_j are specified according to [6] in terms of the elements of Eshelby's tensor, the particle volume fraction within the particulate matrix f_2', and bulk and shear moduli of the matrix and particles (note that in [6] the matrix and particles are identified by subscript "0" and "1", respectively; accordingly, we can use equations in [6] with $0 \to 1$, $1 \to 2$). Of course, the coefficients $F_{ij}(\theta)$ are immediately available.

The principal stresses can now be determined from

$$\begin{vmatrix} \sigma_{11},_m - \sigma & \sigma_{12},_m & 0 \\ \sigma_{12},_m & \sigma_{22},_m - \sigma & 0 \\ 0 & 0 & \sigma_{33},_m - \sigma \end{vmatrix} = 0 \tag{7}$$

in the form

$$\sigma_{1},_{2} = \hat{\sigma}_{11}\left[\frac{F_{11}(\theta)+F_{22}(\theta)}{2} \pm \sqrt{(F_{11}(\theta)-F_{22}(\theta))^2 + 4F_{12}^2(\theta)}\right] = \hat{\sigma}_{12}F_{1},_{2}(\theta)$$

$$\sigma_3 = \hat{\sigma}_{11}F_{33}(\theta) \tag{8}$$

The maximum principal stress criterion yields the tensile strength of the particulate matrix:

$$s_{pm},_T = s_{mT}\min\{F_1^{-1}(\theta),\ F_2^{-1}(\theta),\ F_{33}^{-1}(\theta)\} \tag{9}$$

The von Mises strength criterion yields

$$s_{pm},_T = \sqrt{2}s_{mT}\left\{[F_1(\theta)-F_2(\theta)]^2 + [F_1(\theta)-F_{33}(\theta)]^2 + [F_2(\theta)-F_{33}(\theta)]^2\right\}^{-1} \tag{10}$$

Either one uses the failure criterion (9) or (10), it is necessary to check all values of $0 \le \theta \le \pi/2$ since it is unpractical to analytically determine the angular coordinate corresponding to the onset of failure (the smallest value of the stress given by (9) or (10) corresponds to the actual strength).

The analysis of the axial compressive strength is quite similar, i.e. we can use (9) or (10) where $s_{pm},_T$ is replaced with the compressive strength of the particulate matrix $s_{pm},_C$ and s_{mT} is replaced with the compressive strength of the matrix material s_{mC}.

Consider now the shear strength of the particulate matrix subject to shear stress $\hat{\sigma}_{12}$. Let

$$R = (1 - f_2')(1 - 2S_{1212}) - \frac{G_2}{G_2 - G_1}$$

$$S = \frac{(1 - f_2')(1 - 2S_{1212})\left(1 - \frac{2G_1}{G_{pm}}\right) - \frac{G_2}{G_2 - G_1}}{(1 - f_2')(1 - 2S_{1212}) - \frac{G_2}{G_2 - G_1}} \tag{11}$$

where G_1 and G_2 are the shear moduli of the matrix and inclusions, respectively, and G_{pm} is the shear modulus of the particulate matrix found using the solution in the previous section. Then the stresses in the matrix adjacent to the inclusion can be obtained as

$$\sigma_{11},_m = -\frac{4G_1}{(1-\nu_1)G_{pm}R}\hat{\sigma}_{12}\sin\theta\cos^3\theta = \widetilde{F}_{11}(\theta)\hat{\sigma}_{12}$$
$$\sigma_{22},_m = -\frac{4G_1}{(1-\nu_1)G_{pm}R}\hat{\sigma}_{12}\sin^3\theta\cos\theta = \widetilde{F}_{22}(\theta)\hat{\sigma}_{12}$$
$$\sigma_{33},_m = -\frac{4G_1\nu_1}{(1-\nu_1)G_{pm}R}\hat{\sigma}_{12}\sin\theta\cos\theta = \widetilde{F}_{33}(\theta)\hat{\sigma}_{12} \tag{12}$$
$$\sigma_{12},_m = \hat{\sigma}_{12}\left[S + \frac{4G_1}{(1-\nu_1)G_{pm}R}\sin^2\theta\cos^2\theta\right] = \widetilde{F}_{12}(\theta)\hat{\sigma}_{12} \qquad \sigma_{13},_m = \sigma_{23},_m = 0$$

Note that the solution depends on the modulus of the particulate matrix that is available from the micromechanical solution.

The principal stresses found from (9) are:

$$\sigma_1,_2 = \hat{\sigma}_{12}\left[\frac{\widetilde{F}_{11}(\theta) + \widetilde{F}_{22}(\theta)}{2} \pm \sqrt{\left(\widetilde{F}_{11}(\theta) - \widetilde{F}_{22}(\theta)\right)^2 + 4\widetilde{F}_{12}^2(\theta)}\right] = \hat{\sigma}_{12}\widetilde{F}_1,_2(\theta)$$
$$\sigma_3 = \hat{\sigma}_{12}\widetilde{F}_{33}(\theta) \tag{13}$$

Subsequently, the maximum principal stress criterion or the von Mises criterion yields the shear strength of the particulate matrix in the form (9) and (10), respectively, where $s_{pm},_T$ is replaced with $s_{pm},_S$ (particulate matrix shear strength) and s_{mT} is replaced with the matrix shear strength s_{mS}. As was the case with the tensile or compressive particulate matrix strength, it is necessary to find the smallest stress by varying the values of θ; this result corresponds to the shear strength of the particulate matrix.

2.3. Strength of the fiber-reinforced material with the homogenized matrix

The outline of micromechanical solutions for the strengths in axial and transverse directions as well as the shear strength obtained by assumption that all constituents remain within the linear elastic range and the bonding between the fibers and matrix is not violated was given by Daniel and Ishai [10]. The composite strengths depend on the strength of fibers that we assume known and on the strengths, ultimate strain and the stiffness of the particulate matrix evaluated in the previous sections.

The longitudinal tensile strength s_{1T} depends on the relationship between the ultimate longitudinal tensile strain of fibers $\varepsilon^{u}_{f,l}$ and the ultimate tensile strain of the particulate matrix $\varepsilon^{u}_{pm} = \frac{s_{pm,T}}{E_{pm}}$ (E_{pm} being the elastic modulus of the particulate matrix):

$$\varepsilon^{u}_{f,l} < \varepsilon^{u}_{pm}: \qquad s_{1T} = s_{fT}\left(f_3 + f_{pm}\frac{E_{pm}}{E_3} \right)$$
$$\varepsilon^{u}_{f,l} > \varepsilon^{u}_{pm}: \qquad s_{1T} = s_{pm,T}\left(f_3\frac{E_3}{E_{pm}} + f_{pm} \right) \tag{14}$$

where s_{fT} is the tensile strength of isotropic fibers and E_3 is their modulus of elasticity.

The modes of failure of a unidirectional fiber-reinforced composite subject to longitudinal compression include fiber microbuckling in either extensional or shear mode and shear failure. The microbuckling failure modes occur at the following values of the applied compressive stress:

$$s'_{1C} = \min\left\{ 2f_3\sqrt{\frac{E_{pm}E_3f_3}{3f_{pm}}},\ \frac{G_{pm}}{f_{pm}} \right\} \tag{15}$$

where G_{pm} is the shear modulus of the particulate matrix.

The shear failure mode of a longitudinally compressed fiber-reinforced material occurs at the stress

$$s''_{1C} = 2s_{fs}\left(f_3 + f_{pm}\frac{E_{pm}}{E_3} \right) \tag{16}$$

where s_{fs} is the shear strength of the fiber.

The longitudinal compressive strength is now found as $s_{1C} = \min\{s'_{1C},\ s''_{1C}\}$. The analysis can also account for the effect of fiber misalignment as discussed in [10].

The transverse tensile failure of fiber-reinforced composites can be predicted accounting for the stress or strain concentration factor and for residual stresses [10]. The stress concentration factor for a square array of fibers can be obtained in terms of the properties of the particulate matrix and fibers as

$$k = \frac{1 - f_3\left(1 - E_{pm}/E_3\right)}{1 - \sqrt{4f_3/\pi}\left(1 - E_{pm}/E_3\right)} \tag{17}$$

Subsequently, the maximum principal stress criterion yields $s_{2T} = \left(s_{pmT} - \sigma_{pm,res}\right)/k$ where $\sigma_{pm,res}$ is the maximum radial residual stress in the particulate matrix. The latter stress can be found using a concentric cylinder model subject to a uniform temperature, the inner cylinder being the fiber, surrounded with a cylindrical layer of the particulate matrix that is in turn surrounded with the fiber-reinforced medium. An alternative formulation employing the maximum principal strain criterion is also available from [10] using the properties of the particulate matrix and the maximum residual radial strain.

The compressive strength of a fiber-reinforced composite is the lesser failure stress corresponding to a number of possible scenarios, including interfacial shear failure, debonding and fiber crushing. The most typical mode of failure being the compressive matrix failure, the strength is determined as $s_{2C} = \left(s_{pm},_{C} + \sigma_{pm,res}\right)/k$.

In-plane shear failure occurs as a result of the interfacial shear stress concentration; the stress concentration factor k_{sh} available from (17) by replacing the moduli of elasticity with the shear moduli of the corresponding phases. Then the shear strength is $s_{12} = s_{pm},_{S}/k_{sh}$.

Note that the strengths of the fiber-reinforced material can be specified only upon the conclusion of the micromechanical stiffness analysis presented above since they depend on the stiffness of the particulate matrix.

3. NUMERICAL EXAMPLES

The effectiveness of embedding stiff particles within a matrix of fiber-reinforced composites was shown on the example of a glass/epoxy material with spherical particles within the matrix. The properties of the constituent materials were taken as in [2]: $E_1 = 3.12GPa,\ \nu_1 = 0.38,\ E_2 = E_3 = 76.0GPa,\ \nu_2 = \nu_3 = 0.25$. The tensile stress ratio $k_{pm} = \sigma_{11},_{m}(\max)/\hat{\sigma}_{11}$ as a result of uniaxial tension is shown in Fig. 1. The case of $f_2' = 0$ corresponds to a single particle embedded within the matrix, while large values of the particle volume fraction account for multiple inclusions. The stress ratio reaches a maximum in the case of a single particle as was also the case in [6]. The tensile stress ratio at the composite level, i.e. $\sigma_{11,m}(\max) \big/ \sigma_{11}^{0}$ (rather than the stress applied to the particulate matrix as was the case in Fig. 1) is shown in Fig. 2.

The beneficial effect of adding particles on the longitudinal and transverse stiffness of the fiber-reinforced material is reflected in Fig. 3. The longitudinal stiffness of the material with the homogeneous particulate matrix was determined by the rule of mixtures. The transverse stiffness was determined by the Halpin-Tsai model with the curve fitting parameter equal to $\xi = 1$ and $\xi = 2$ (typical range of this parameter). As follows from Fig. 3, even a modest amount of particles added to the matrix can significantly enhance the transverse stiffness, although the effect on the longitudinal stiffness is less pronounced.

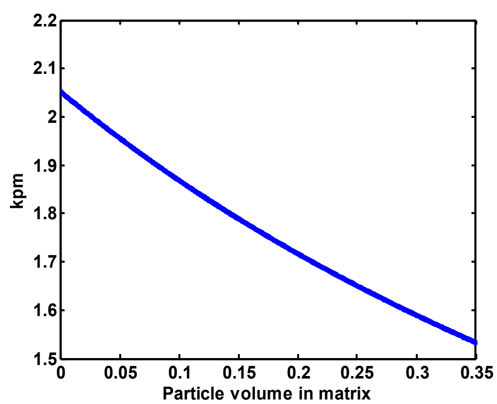


Fig. 1. Tensile stress ratio (k_{pm}) in the particulate matrix subject to uniaxial tension.

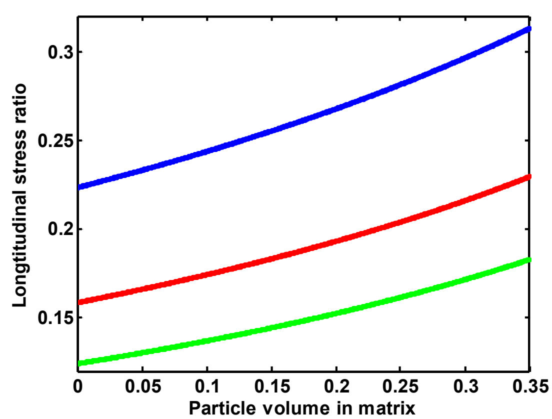
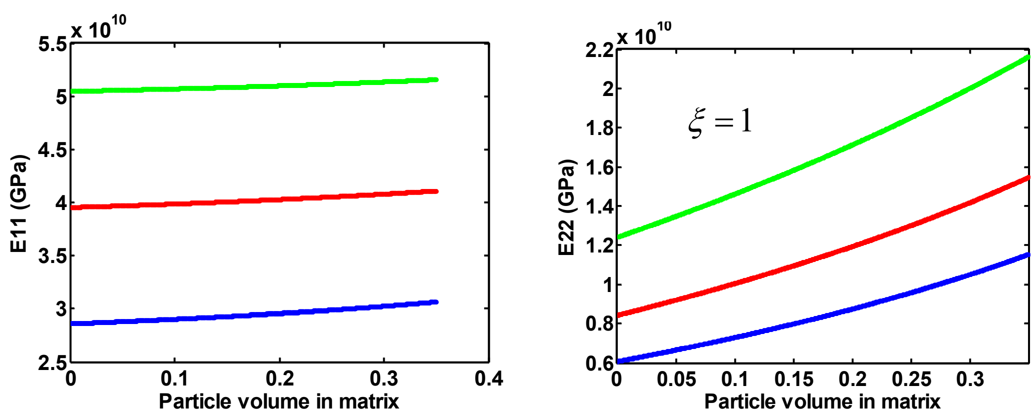


Fig. 2. Tensile stress ratio in fiber-reinforced particulate-matrix composite (blue: f_3=0.35, red: f_3=0.50, green: f_3=0.65).



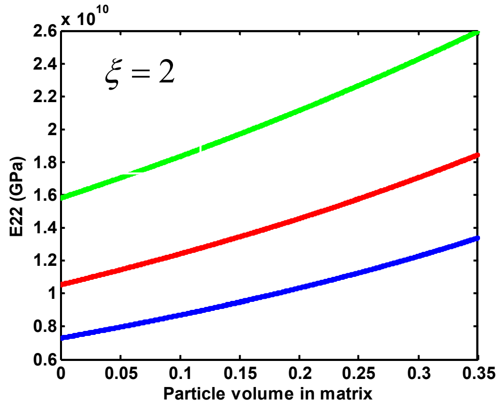


Fig. 3. Effect of particle volume fraction in particulate matrix on longitudinal and transverse stiffness of fiber-reinforced composite (blue: f_3=0.35, red: f_3=0.50, green: f_3=0.65).

The effect of particles on the longitudinal tensile strength of the composite material is reflected in Fig. 4 where $R = {}^{s_{1T}}\!/_{s_{fT}}$ for the case where $\varepsilon^u_{f,l} < \varepsilon^u_{pm}$. As is obvious from Fig. 4, adding glass particles to the matrix has a relatively small effect on the longitudinal strength of the material. Predictably, the situation is different in the case of transverse strength since it is highly dependent on the strength of the particulate matrix. As is shown in Fig. 5, the effect of adding particles on the transverse strength of the composite is much more pronounced than that on its longitudinal strength. This is expected since the contribution of the matrix to the transverse strength is higher than that of to the longitudinal strength.

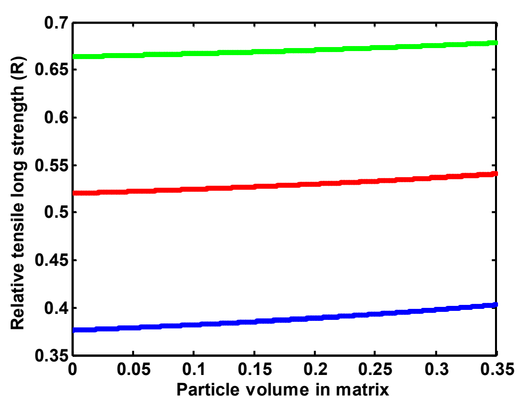


Fig. 4. The effect of particles on the tensile longitudinal strength of fiber-reinforced particulate-matrix composite (blue: f_3=0.35, red: f_3=0.50, green: f_3=0.65).

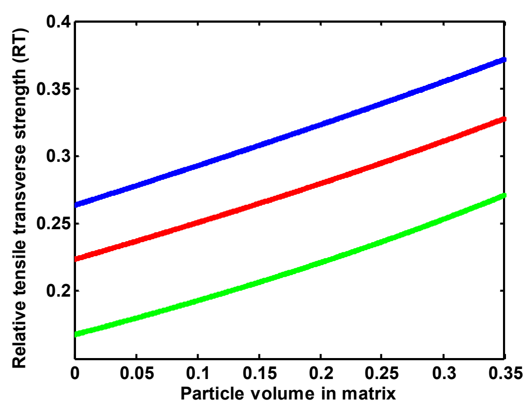


Fig. 5. The effect of particles on the tensile transverse strength of fiber-reinforced particulate-matrix composite (blue: f_3=0.35, red: f_3=0.50, green: f_3=0.65).

4. CONCLUSIONS

The paper illustrates a two-step approach to the strength and stiffness analyses of fiber-reinforced, particulate-matrix composites. The solution is obtained by a generalization of available micromechanical solutions to three-phase materials. The strength and stiffness of the particulate matrix are specified first, followed with the analysis of the properties of a fiber-reinforced material incorporating the already homogenized matrix.

As follows from the numerical analysis, the addition of stiff particles to the matrix results in a significant enhancement of transverse strength and stiffness. The benefits of added particles are less obvious for longitudinal strength and stiffness. These results are due to a relatively larger contribution of the matrix to the transverse properties of the fiber-reinforced material.

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REFERENCES

1. Birman, V. and Byrd, L.W., "Modeling and analysis of functionally graded materials and structures," *Applied Mechanics Reviews*, 2007; 60: 195-216.
2. Genin, G.M. and Birman, V., "Micromechanics and structural response of functionally graded, particulate-matrix, fiber-reinforced composites," *International Journal of Solids and Structures*, In Press.
3. Tucker, III., C.L. and Liang, E., "Stiffness predictions for unidirectional short-fiber composites: Review and Evaluation," *Composites Science and Technology*, 1999; 59: 655-671.
4. Hu, G.K. and Weng, G.J., "The connections between the double-inclusion model and the Ponte Castaneda-Willis, Mori-Tanaka, and Kuster-Toksoz models," *Mechanics of Materials*, 2000; 32: 495-503.
5. Benveniste, Y., "A new approach to the application of Mori-Tanaka theory in composite materials," *Mechanics of Materials*, 1987; 6: 147-157.
6. Tandon, G.P. and Weng, G.J., "Stress distribution in and around spheroidal inclusions and voids at finite concentration," *Journal of Applied Mechanics*, 1986; 53: 511-518.
7. Luo, H.A. and Weng, G.J., "On Eshelby's S-tensor in a three-phase cylindrically concentric solid, and the elastic moduli of fiber-reinforced composites," *Mechanics of Materials*, 1989; 8: 77-88.
8. Eshelby, J.D., "The determination of the elastic field of an ellipsoidal inclusion, and related problems," *Proceedings of the Royal Society, London*, Series A, 1957; 241:376-396.
9. Kakavas, P.A. and Kontoni, D-P., "Numerical investigation of the stress field of particulate reinforced polymeric composites subjected to tension," *International Journal of Numerical Methods in Engineering*, 2005; 65: 1145-1164.
10. Daniel, I.M. and Ishai, O., Engineering Mechanics of Composite Materials, Second Edition, *Oxford University Press*, New York, 2006.