ANALYSIS OF CYLINDRICAL SANDWICH SHELLS USING A MESHLESS METHOD AND AN OPTIMIZATION TECHNIQUE

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ABSTRACT

The purpose of this work is to use a meshless collocation method with multiquadric radial basis functions (RBFs), the higher order shear deformation theory presented by Khare et al. [1], and optimal values of the shape parameter in the RBFs to analyze static deformations of sandwich cylindrical shells.
An optimization technique based on a statistical method is used to choose a shape parameter c on the interpolating multiquadric radial basis function used by the meshless method. This parameter plays a decisive role in the quality of the solution of the boundary value problem and is usually chosen in a trial error basis. The optimization technique here presented allows to obtain a quasi user-independent shape parameter. Although the technique still requires some input by the user, results are encouraging, with the errors produced by the optimal shape parameter being lower than the ones produced by a user defined shape parameter.

1. INTRODUCTION

In this paper we use a higher-order shear deformation theory with a meshless numerical method (radial basis function collocation method) for the modelling of cylindrical sandwich shells. The introduction of transverse shear and normal stresses represent an improvement to classical theories. Classical theories developed for thin elastic shells are mostly based on the Love-Kirchhoff assumptions. This theory considers that straight lines normal to the undeformed middle surface remain straight and normal to the deformed middle surface; that the normal stresses perpendicular to the middle surface can be neglected in the stress-strain relations and the transverse displacement is independent of the thickness coordinate. Therefore transverse shear strains are neglected as reported in surveys of classical shell theories by Naghdi [2] and Bert [3, 4]. These theories are expected to produce accurate results when the side-to thickness ratio (a/h) is large or when material anisotropy is low. The application of such theories to thick or moderately thick or laminated composite shells can lead to serious errors in terms of deflection or stresses.
The HOST9 higher order shell deformation theory presented by Khare et al. [1] is in this paper reduced to seven degrees of freedom, and produces accurate results in the modelling of sandwich shells. The theory accounts for transverse shear stresses and assumes a parabolic distribution of transverse shear strain through the shell thickness.
The equilibrium equations are derived considering the following displacement field (u,v,w):

$$u(x,y,z) = u_0(x,y) + z\phi_x(x,y) + z^3\phi_x^*(x,y) \quad (1)$$

$$v(x,y,z) = v_0(x,y) + z\phi_y(x,y) + z^3\phi_y^*(x,y) \quad (2)$$

$$w(x,y,z) = w_0(x,y) \tag{3}$$

In this paper we compare the analytical solution for several shells as presented in [1] with the solution obtained with the meshless multiquadric radial basis functions method. The meshless multiquadric method is well known for solving systems of partial differential equations with excellent accuracy [5,6]. However, it has the problem of the choice of an adequate shape parameter for the multiquadric interpolation function. We use a statistical technique based on Rippa's algorithm [7] that overcomes the choice of the shape parameter to the simple indication of a user-defined interval. This collocation method, in a general way, can use a variety of radial basis functions as interpolation functions, such as those in equations (4)-(7). Figure 1 shows a plot of these functions.

$$g(r,c) = \sqrt{(c^2 + r^2)}; \quad \text{multiquadric} \tag{4}$$

$$g(r,c) = 1/\sqrt{(c^2 + r^2)}; \quad \text{inverse multiquadric} \tag{5}$$

$$g(r,c) = e^{-cr^2}; c > 0; \quad \text{gaussian} \tag{6}$$

$$g(r) = r^2 \log(r); \quad \text{thin plate spline} \tag{7}$$

Radial basis functions depend on a distance *r* between points in a grid, and some may depend on a shape parameter, *c*.

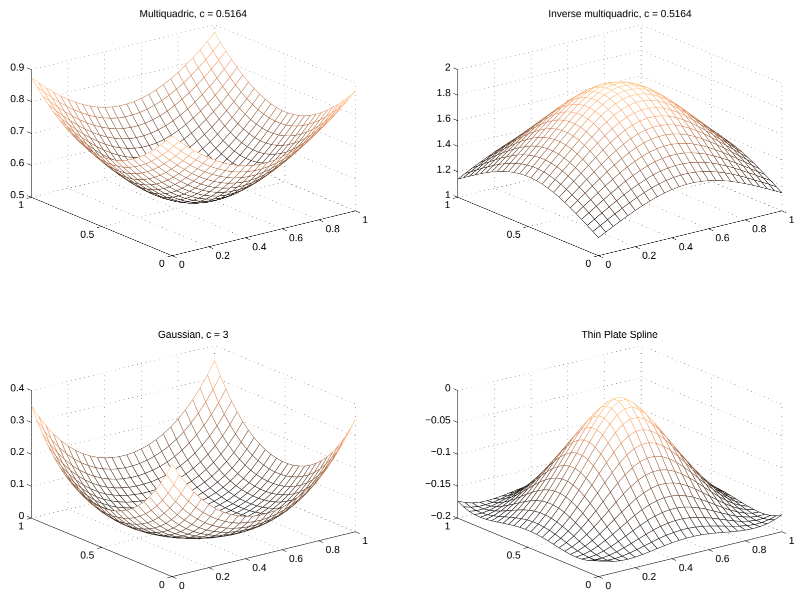

Figure1: Examples of radial basis functions in 2D.

The radial basis function used in this paper is the multiquadric function (equation (4)). The multiquadric function depends on a shape parameter, *c*, that works as a fine tuning, producing a smooth curved surface or a hat shaped function, as seen in Figure 2.
Several proposals for the choice of a adequate shape parameter can be found in the papers of Hardy [8], Franke [9], Kansa [5, 6], Fasshauer [10]. All these proposals are somehow related with the number of points in the grid and the distance between those points (Table 1). However, according to Rippa [7], the shape parameter should depend

on many other factors, such as: number of grid points, distribution of points, interpolation function *g(r,c)*, conditioning number of matrix and computer precision.

Reference	Shape parameter, c
Hardy, 1971 [8]	$c = 0.81d$
Franke, 1982 [9]	$c = 1.25D/\sqrt{N}$
Kansa, 1990 [5, 6]	$c^2 = c_{\min}^2 (c_{\max}^2 / c_{\min}^2)^{(j-1)/(N-1)}$
Fasshauer, 2002 [10]	$c = 2/\sqrt{N}$
	d,D-distances; j,N-number of points in grid

Table1: Some proposals for the choice of a shape parameter for the multiquadric function.

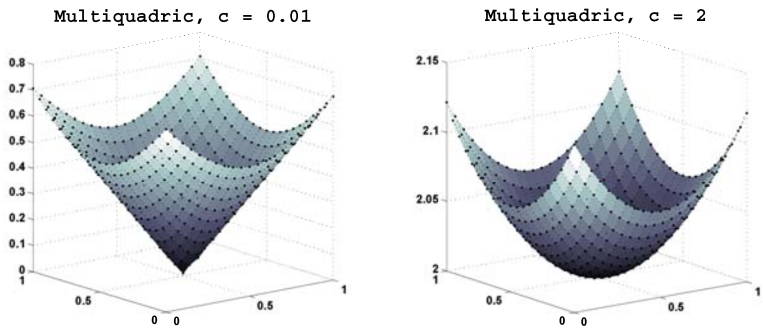

Figure 2. The multiquadric function, with different shape parameters, *c*.

The optimization technique proposed in this paper is based on a statistical procedure and takes into account the recommendations of Rippa [7]. The cross validation leave-one-out technique from regression analysis is applied to a user-defined interval. The optimal value for the shape parameter will be searched within that interval, and therefore the optimization technique still depends on a user input, but of easier choice.

2. THE MULTIQUADRIC MESHLESS METHOD

Most of the spatial discretization techniques used in engineering have been based on finite differences and finite elements. In the present study we use the multiquadric radial basis function collocation method. This truly meshless technique is insensitive to spatial dimension, considers only a cloud of nodes for the spatial discretization of both the problem domain and the boundary. For the collocation technique of differential equations and boundary conditions we use the direct collocation method, proposed by Kansa [5, 6].

Consider the generic boundary value problem with a domain Ω with boundary $\partial\Omega$, and linear differential operators *L* and *B*.

$$Lu(x) = f(x), x \in \Omega \subset \mathbb{R}^n; \quad Bu|_{\partial\Omega} = q \tag{8}$$

The function **u(x)** is approximated by:

$$u\ ;\ \theta\!/\!\sigma = \sum_{j=1}^{N} \alpha_j g_j \tag{9}$$

were α_j are parameters to be determined after the collocation method is applied and *N* is the total number of points. We consider a global collocation method where the linear operators *L* and *B* acting in domain $\Omega \setminus \partial\Omega$ and boundary $\partial\Omega$ define a set of global equations in the form:

$$\begin{pmatrix} L_{ii} & L_{ib} \\ B_{bi} & B_{bb} \end{pmatrix} \begin{pmatrix} \alpha_i \\ \alpha_b \end{pmatrix} = \begin{pmatrix} f_i \\ q_b \end{pmatrix}; \quad or \quad \left[L\right]\left[\alpha\right] = \left[\lambda\right] \tag{10}$$

where *i* and *b* denote domain and boundary nodes, respectively; f_i and q_b are external conditions in domain and boundary (in shells in bending these can be external forces).

The function g represents a radial basis function. In the present case, we choose the multiquadric function, defined as:

$$g(r,c) = \left(r^2 + c^2\right)^{1/2} \tag{11}$$

were *r* is the euclidian distance between two distinct nodes and *c* is a shape parameter that improves the function surface so that convergence gets faster [5, 6]. Other radial basis functions could be used (gaussians, splines, etc). However, multiquadrics proved to be excellent for global, smooth, boundary-value problems, such as the problem of shells in bending [11, 12].

3. CROSS VALIDATION OPTIMIZATION TECHNIQUE

An optimal shape parameter *c* can be obtained for an interpolation problem $A\alpha = f$, $A = g(\| x_j - x_i, c \|)$ by the leave-one-out cross validation technique in regression analysis. The problem can be formulated as finding *c* in order to minimize a cost function given by the norm of an error vector *E(c)* with components:

$$E_i(c) = f_i - \sum_{j=1,\, j\neq i}^{N} \alpha_j^{(i)} g(\| x_j - x_i, c \|) \tag{12}$$

Here $\sum_{j=1,\, j\neq i}^{N} \alpha_j^{(i)} g(\| x_j - x_i, c \|)$ is the function value predicted at the *i*-th data point using RBF interpolation based on a set of data that excludes the *i*-th point. A more efficient algorithm, from a computational point of view, is given by the following formula [7, 13]:

$$E_i(c) = \frac{\alpha_i}{A^{-1}_{i,i}} \tag{13}$$

where α_i is the *i*-th coefficient for the full interpolation problem and $A^{-1}_{i,i}$ is the i-th diagonal element of the inverse of the corresponding interpolation matrix *A*. In the case of our boundary value problem, the error to be minimized is a residual error, of the form [14]:

$$E_i(c) = \lambda_i - \sum_{j=1, j\neq i}^{N} \alpha_j^{(i)} Lg(\| x_j - x_i, c \|) \tag{14}$$

Now the generalization of the cross-validation algorithm is straightforward. Our BVP is given by equation (6). We can use the following formula that is analogous to (9):

$$E_i(c) = \frac{\alpha_i}{L_{i,i}^{-1}} \tag{15}$$

where α_i is the i-th coefficient for the full collocation problem (6) and $L_{i,i}^{-1}$ i-th diagonal element of the inverse of the corresponding collocation matrix *L*. Having the cost function, we use the MATLAB function **fminbnd** to find a local minimum.

4. EQUILIBRIUM EQUATIONS AND BOUNDARY CONDITIONS

The equilibrium equations are derived considering following the displacement field (u,v,w) in equations (1)-(3). For small displacements and assuming $\frac{h}{R_x}, \frac{h}{R_y} = 1$, the strain-displacements relations are given by:

$$\begin{gathered}
\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{w}{R_x}; \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{w}{R_y}; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; \\
\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \frac{v}{R_y}; \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} - \frac{u}{R_x};
\end{gathered} \tag{16}$$

Replacing the expressions for the displacement field (u,v,w) in equation (16), the strain displacements relations are now expressed by:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \\ \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \\ \gamma_{yz}^{(1)} \\ \gamma_{xz}^{(1)} \end{Bmatrix} + z^2 \begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \gamma_{xy}^{(2)} \\ \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \\ \gamma_{yz}^{(3)} \\ \gamma_{xz}^{(3)} \end{Bmatrix} \tag{17}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{w_0}{R_x} \\ \frac{\partial v_0}{\partial y} + \frac{w_0}{R_y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \\ \phi_y + \frac{\partial w_0}{\partial y} - \frac{v_0}{R_x} \\ \phi_x + \frac{\partial w_0}{\partial x} - \frac{u_0}{R_y} \end{Bmatrix} + z \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \\ -\frac{\phi_y}{R_y} \\ -\frac{\phi_x}{R_x} \end{Bmatrix} + z^2 \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 3\phi_y^* \\ 3\phi_x^* \end{Bmatrix} + z^3 \begin{Bmatrix} \frac{\partial \phi_x^*}{\partial x} \\ \frac{\partial \phi_y^*}{\partial y} \\ \frac{\partial \phi_x^*}{\partial y} + \frac{\partial \phi_y^*}{\partial x} \\ -\frac{\phi_y^*}{R_y} \\ -\frac{\phi_x^*}{R_x} \end{Bmatrix} \tag{18}$$

By neglecting transverse normal stress, σ_z, the stress-strain relations in the local (material) cartesian system can be obtained as:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \tag{19}$$

where subscripts 1 and 2 are respectively the fiber and the normal to fiber inplane directions, and 3 is the direction normal to the plate. The reduced stiffness components, Q_{ij} are given by:

$$\begin{gathered} Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}; \quad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}; \quad Q_{12} = \nu_{21} Q_{11}; \\ Q_{33} = G_{12}; \quad Q_{44} = G_{23}; \quad Q_{55} = G_{31}; \quad \nu_{21} = \nu_{12}\frac{E_2}{E_1} \end{gathered} \tag{20}$$

in which $E_1, E_2, \nu_{12}, \nu_{21}, G_{12}, G_{23}, G_{31}$ are material properties of the lamina.
The equilibrium equations are then obtained using the principle of virtual work,

$$\delta\Pi = \int_x \int_y \int_z (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz} - q\delta w_0)\, dz\, dy\, dx = 0 \tag{21}$$

producing seven equations of equilibrium:

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \frac{Q_{xz}}{R_x} = 0 \tag{22}$$

$$\frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} + \frac{Q_{yz}}{R_y} = 0 \tag{23}$$

$$\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} - \frac{N_{xx}}{R_x} - \frac{N_{yy}}{R_y} = q \tag{24}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{xz} + \frac{K_{xz}}{R_x} = 0 \tag{25}$$

$$\frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_{yz} + \frac{K_{yz}}{R_y} = 0 \tag{26}$$

$$\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} - 3T_{xz} + \frac{U_{xz}}{R_x} = 0 \tag{27}$$

$$\frac{\partial S_{yy}}{\partial y}+\frac{\partial S_{xy}}{\partial x}-3T_{yz}+\frac{U_{yz}}{R_y}=0 \quad (28)$$

where the stress resultants and moments are expressed as

$$\begin{Bmatrix} N_{ij} \\ M_{ij} \\ S_{ij} \end{Bmatrix}=\int_{-h/2}^{h/2}\sigma_{ij}\begin{Bmatrix} 1 \\ z \\ z^3 \end{Bmatrix}dz;\quad \begin{Bmatrix} Q_{iz} \\ K_{iz} \\ T_{iz} \\ U_{iz} \end{Bmatrix}=\int_{-h/2}^{h/2}\tau_{iz}\begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix}dz;\quad i,j=x,y \quad (29)$$

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{S\} \end{Bmatrix}=\begin{bmatrix} [A] & [B] & [E] \\ [B] & [F] & [G] \\ [E] & [G] & [H] \end{bmatrix}\begin{Bmatrix} \{\varepsilon^{(0)}\} \\ \{\varepsilon^{(1)}\} \\ \{\varepsilon^{(3)}\} \end{Bmatrix};\quad \begin{Bmatrix} \{Q\} \\ \{K\} \\ \{T\} \\ \{U\} \end{Bmatrix}=\begin{bmatrix} [I] & [K] & [L] & [M] \\ [K] & [N] & [O] & [P] \\ [L] & [O] & [Q] & [R] \\ [M] & [P] & [R] & [S] \end{bmatrix}\begin{Bmatrix} \{\gamma^{(0)}\} \\ \{\gamma^{(1)}\} \\ \{\gamma^{(2)}\} \\ \{\gamma^{(3)}\} \end{Bmatrix} (30)$$

The stiffness components in equation (30) are given by:

$$(A_{ij},B_{ij},E_{ij},F_{ij},G_{ij},H_{ij})=\int_{-h/2}^{h/2}Q(i,j)(1,z,z^3,z^2,z^4,z^6)\,dz;\quad i,j=1,2,3$$

$$(I_{ij},K_{ij},L_{ij},M_{ij},N_{ij},O_{ij},P_{ij},Q_{ij},R_{ij},S_{ij})=\int_{-h/2}^{h/2}Q(i,j)(1,z,z^2,z^3,z^2,z^3,z^4,z^4,z^5,z^6)\,dz;\quad i,j=4,5$$

(31)

The boundary conditions for a simply supported shell for a border along the x axis are:

$$u_0=0;\quad N_{xx}=0;\quad w_0=0;\quad \phi_x=0;\quad M_{yy}=0;\quad \phi_x^*=0;\quad S_{yy}=0 \quad (32)$$

5. NUMERICAL EXAMPLES

In this section we analyze cylindrical sandwich shells (face sheet/ core/ face sheet) of side *a*, thickness *h* and curvature radius *R*. All the shells have simply supported boundary conditions and are subjected to a sinusoidal load $q=q_0\sin(\pi a)\cos(\pi a)$. The results obtained with the present method are compared with an analytical solution of Khare et al. [1]. To evaluate the results, a relative error is calculated by:

$$\text{error}(\%)=\left|\frac{\bar{w}-\text{analytical solution}[1]}{\text{analytical solution}[1]}\right|\times 100 \quad (33)$$

Cylindrical sandwich shells with different thickness/side length and radius/side length ratios are analysed. The mechanical properties of the sandwich layer are given as:

Face sheets: $E_x=25;\quad E_y=1;\quad G_{xz}=0.5;\quad G_{xy}=0.5;\quad G_{yz}=0.2;\quad \nu_{xy}=0.25; h_f=0.1h$

Core: $E_x=0.04;\quad E_y=0.04;\quad G_{xz}=0.06;\quad G_{xy}=0.016;\quad G_{yz}=0.06;\quad \nu_{xy}=0.25; h_f=0.8h$

The results for the central deflection are normalized by:

$$\bar{w} = \frac{100 w E_y h^3}{q a^4} \quad (34)$$

Two tables are presented. The first one (Table 2) is produced with a user-defined shape parameter of the form $c=2/\sqrt{n}$, being n the number of points/side. Table 3 is obtained using the optimization technique, within the interval]0.01-2[. The value of the optimized shape parameter is indicated by $c=c_{opt}$.

The relative error in percentage is indicated in tables 2 and 3. For the majority of the examples, an improvement of the relative error is observed when using the optimization technique. There are some isolated cases in which the optimization technique fails to produce a smaller error, for example, *R/h=20* with *n=13*.

h/a	R/h	Number of points/side, *n*							HOST9[1]
		9	11	13	15	17	19	21	
0.01	100	0.0652	0.0815	0.0941	0.1028	0.1084	0.1118	0.1140	0.1180
	Error (%)	44.7	30.9	20.3	12.9	8.1	5.3	3.4	
	50	0.0216	0.0243	0.0267	0.0287	0.0301	0.0310	0.0315	0.0326
	Error (%)	33.7	25.5	18.1	12.0	7.7	5.0	3.4	
	20	0.0052	0.0053	0.0054	0.0054	0.0054	0.0053	0.0053	0.0052
	Error (%)	0.9	1.7	3.0	3.2	2.7	2.1	1.5	
0.1	100	2.0698	2.0776	2.0811	2.0829	2.0838	2.0843	2.0846	2.0853
	Error (%)	0.7	0.4	0.2	0.1	0.07	0.049	0.03	
	50	2.0594	2.0729	2.0791	2.0821	2.0837	2.0846	2.0852	2.0867
	Error (%)	1.3	0.7	0.4	0.2	0.1	0.09	0.07	
	20	1.9909	2.0403	2.0640	2.0760	2.0825	2.0862	2.0883	2.0958
	Error (%)	5.0	2.6	1.5	0.9	0.6	0.5	0.4	

Table 2: Central normalized displacement, with $c = 2/\sqrt{n}$, for a cylindrical sandwich shell.

6. CONCLUSIONS

The higher order shear deformation theory of Khare et al. [1] was used to compute static transverse central deflection of cylindrical sandwich shells. To solve the equilibrium equations, the meshless multiquadric method is used, with an optimization technique based on cross validation analysis. The results obtained with the optimization technique are generally better, when compared with a user defined shape parameter. Although the cross validation technique increases the computational time of the problem, a smaller error can be achieved with fewer number of grid points. This can be particularly important when solving large engineering problems.

h/a	R/h	Number of points/side, *n*							HOST9[1]
		9	11	13	15	17	19	21	
0.01	100	0.1095	0.1171	0.1075	0.1172	0.1081	0.1179	0.1179	0.1180
	c_opt	1.6166	1.5302	0.7701	0.9832	0.4798	1.2399	1.5302	
	Error (%)	7.2	0.8	8.9	0.7	8.4	0.09	0.09	
	50	0.0180	0.0304	0.0309	0.0305	0.0317	0.0312	0.0319	0.0326
	c_opt	1.0604	1.2399	0.8929	0.6592	0.6468	0.4798	0.4798	
	Error (%)	44.8	6.7	5.4	6.4	2.7	4.4	2.2	
	20	0.0052	0.0052	0.0078	0.0052	0.0052	0.0053	0.0053	0.0052
	c_opt	1.5302	1.0604	0.7701	1.1679	1.2399	0.5805	0.4798	
	Error (%)	0.2	1.1	49.3	0.05	0.06	1.2	1.1	
0.1	100	2.0831	2.0850	2.0850	2.0847	2.0850	2.0850	2.0851	2.0853
	c_opt	1.3974	1.4208	1.2399	0.8034	0.8930	0.7180	0.6374	
	Error (%)	0.1	0.02	0.02	0.03	0.01	0.01	0.01	
	50	2.0841	2.0857	2.0860	2.0858	2.0858	2.0859	2.0861	2.0867
	c_opt	1.5302	1.3792	1.3261	0.9135	0.7701	0.6421	0.7701	
	Error (%)	0.1	0.05	0.03	0.04	0.04	0.04	0.03	
	20	2.0788	2.0868	2.0895	2.0905	2.0901	2.0917	2.0904	2.0958
	c_opt	1.3968	1.1713	1.0039	0.8666	0.6709	0.7701	0.5198	
	Error (%)	0.8	0.4	0.3	0.3	0.3	0.2	0.3	

Table 3: Central normalized displacement, with an optimal *c*, for a cylindrical sandwich shell.

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