APPROXIMATE METHOD OF PREDICTING PLY CRACK FORMATION IN GENERAL SYMMETRIC LAMINATES SUBJECT TO BIAXIAL LOADING AND BENDING

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ABSTRACT

Ply cracking is a very important damage mode for laminated composite materials that has been the subject of intensive study over many years, especially cross-ply laminates. In practical applications, laminates are usually more complex than cross-plies, and they are subject to complex loading that involves both in-plane and out-of-plane deformation, in addition to residual stresses arising from thermal expansion mismatch effects between the plies. Based on a methodology that is able to predict ply cracking for: i) a multiple-ply cross-ply laminate subject to combined biaxial in-plane loading and biaxial bending, and ii) a general symmetric laminate subject to combined in-plane biaxial loading and in-plane shear loading, where thermal residual stresses can be present, this paper will attempt to combine the models developed for these situations so that ply cracking in general symmetric laminates subject to combined in-plane biaxial *and* out-of-plane biaxial bending can be treated. It is first assumed that laminates have at least one 90^o ply, and that ply cracking occurs only in some or all of the 90^o plies. The off-axis plies in the laminate, which are uncracked, are first homogenised into an orthotropic effective medium. The next step is to apply the homogenised properties to the 0^o plies of an equivalent hybrid cross-ply laminate that is then subject to progressive loading involving any combination of in-plane biaxial loading and out-of-plane bending. The homogenisation approach is tested, for the case of uniaxial in-plane loading, by comparing predictions of the homogenised model with those of an existing model that allows for the presence of all the off-axis plies. For more complex loading states, an example will be given of a prediction of the progressive growth of ply cracking, and of laminate stress-strain behaviour.

1. INTRODUCTION

As many structures experience some form of bending deformation during service it is vital that damage formation in the presence of bending is well understood. Of particular relevance to the performance of structures is the prediction of the occurrence of microstructural damage in complex loading modes where out-of-plane bending modes of deformation occur in conjunction with in-plane biaxial and through-thickness loading. While a great deal of research has been devoted to the case of ply cracking in cross-ply laminates subject only to in-plane deformations, the practically important case of out-of-plane bending has received much less attention (e.g. [1-8]). One objective of this paper is to summarise the important results that have been derived using an energy balance method for predicting the conditions for the steady state growth of ply cracks in a cross-ply laminate subject to bending and thermal residual stresses. A second objective is to indicate how the methodology for ply crack formation can be developed without a detailed analysis of the stress transfer that is in fact needed only to estimate the thermoelastic constants of a damaged laminate. The anticlastic bending typical of deformed laminates is taken into consideration as are thermal residual stresses.

2. PREDICTING PLY CRACKING DURING BEND DEFORMATION

2.1 Geometry and loading conditions

A general balanced laminate, that might not be symmetric, of length 2L, width 2W and total thickness 2h consisting of perfectly bonded anisotropic layers is considered within a Cartesian coordinate system. The x-direction is taken as the through-thickness direction of the laminate, the y-direction is taken as the axial (longitudinal) direction and the z-direction as the in-plane transverse direction. The origin is located on one of the laminate faces. In order to represent the applied loading that leads to biaxial bending of the laminate for the axial and transverse directions, it is assumed that the edge boundary conditions for the displacement components v and w are of the form

$$v = \begin{cases} L(\varepsilon + \hat{\varepsilon}\, x) & \text{on } \ y = L \\ -L(\varepsilon + \hat{\varepsilon}\, x) & \text{on } \ y = -L \end{cases} \qquad w = \begin{cases} W(\varepsilon_T + \hat{\varepsilon}_T x) & \text{on } \ z = W \\ -W(\varepsilon_T + \hat{\varepsilon}_T x) & \text{on } \ z = -W\,. \end{cases} \tag{1}$$

For infinitesimal deformations, the radii of curvatures of the surface $x = 0$ of the deformed laminate are given by $R_1 = 1/\left|\hat{\varepsilon}\right|$ and $R_2 = 1/\left|\hat{\varepsilon}_T\right|$, so that $\hat{\varepsilon}$ and $\hat{\varepsilon}_T$ are respectively the curvatures of the surface $x = 0$ of the deformed laminate in the axial and transverse directions. The faces of the laminate are assumed to be subject to a uniform applied tensile traction so that

$$\sigma_{xx} = \sigma_t \quad \text{on} \quad x = 0,\ 2h\,. \tag{2}$$

In most practical applications, the applied face tractions would be an applied pressure so that σ_t is negative. If an exact linear elastic analysis of the laminate subject to the boundary conditions (1) and (2) is undertaken, then the applied tractions induced on the laminate edges and displacements induced on the laminate faces will be non-uniform and complex in nature. To deal with this situation an averaging procedure is introduced.

The bending moments per unit area of loading cross-section for the axial and transverse directions are defined respectively by

$$M = \frac{1}{4hW}\int_{-W}^{W}\int_{0}^{2h} (x-h)\, \sigma_{yy}(x,L,z)\,dx\,dz\,, \tag{3}$$

$$M_T = \frac{1}{4hL}\int_{-L}^{L}\int_{0}^{2h} (x-h)\, \sigma_{zz}(x,y,W)\,dx\,dy\,. \tag{4}$$

The moments are taken about the mid-plane of the laminate that might not correspond to the neutral axis if the laminate is unsymmetrical and/or damaged in the region of axial tension. The corresponding effective applied axial and transverse stresses are defined respectively by

$$\sigma = \frac{1}{4hW}\int_{-W}^{W}\int_{0}^{2h} \sigma_{yy}(x,L,z)\,dx\,dz\,, \qquad \sigma_T = \frac{1}{4hL}\int_{-L}^{L}\int_{0}^{2h} \sigma_{zz}(x,y,W)\,dx\,dy\,. \tag{5}$$

The corresponding averaged applied in-plane axial and transverse strains $\widehat{\varepsilon}$ and $\widehat{\varepsilon}_T$ are defined by

$$\left.\begin{aligned} \widehat{\varepsilon} &= \frac{1}{8hLW}\int_{-W}^{W}\int_{0}^{2h} \big[v(x,L,z) - v(x,-L,z)\big]\,dx\,dz\,, \\ \widehat{\varepsilon}_T &= \frac{1}{8hLW}\int_{-L}^{L}\int_{0}^{2h} \big[w(x,y,W) - w(x,y,-W)\big]\,dx\,dy\,, \end{aligned}\right\} \tag{6}$$

and the averaged through-thickness strain ε_t for the laminate is defined by

$$\varepsilon_t = \frac{1}{8hLW}\int_{-W}^{W}\int_{-L}^{L} \big[\, u(2h,y,z) - u(0,y,z) \,\big]\,dy\,dz\,, \tag{7}$$

where u is the x-component of the displacement vector. Substituting the edge boundary conditions (1) into (6) and performing the integrations leads to

$$\widehat{\varepsilon} = \varepsilon + h\,\hat{\varepsilon}\,, \qquad \widehat{\varepsilon}_T = \varepsilon_T + h\,\hat{\varepsilon}_T\,. \tag{8}$$

2.2 Stress-strain relations

In the absence of in-plane shear deformation it is known [2, 3, 6-8] that the effective stress-strain relations, based on the above definitions of applied stress and applied strain for a damaged *cross-ply* laminate having n plies, are of the following form

$$\varepsilon_t = \frac{\sigma_t}{E_t} - \frac{\nu_a}{E_A}\sigma - \frac{\nu_t}{E_T}\sigma_T - \frac{\hat{\nu}_a}{\hat{E}_A}M - \frac{\hat{\eta}_t}{\hat{E}_T}M_T + \alpha_t\Delta T, \tag{9}$$

$$\overset{\frown}{\varepsilon} = -\frac{\nu_a}{E_A}\sigma_t + \frac{\sigma}{E_A} - \frac{\nu_A}{E_A}\sigma_T - \frac{\hat{\nu}_A}{\hat{E}_A}M - \frac{\hat{\eta}_A}{\hat{E}_T}M_T + \alpha_A\Delta T, \tag{10}$$

$$\overset{\frown}{\varepsilon}_T = -\frac{\nu_t}{E_T}\sigma_t - \frac{\nu_A}{E_A}\sigma + \frac{\sigma_T}{E_T} - \frac{\hat{\nu}_T}{\hat{E}_A}M - \frac{\hat{\eta}_T}{\hat{E}_T}M_T + \alpha_T\Delta T, \tag{11}$$

$$\hat{\varepsilon} = -\frac{\hat{\nu}_a}{\hat{E}_A}\sigma_t - \frac{\hat{\nu}_A}{\hat{E}_A}\sigma - \frac{\hat{\nu}_T}{\hat{E}_A}\sigma_T + \frac{M}{\hat{E}_A} - \frac{\hat{\delta}_A}{\hat{E}_A}M_T + \hat{\alpha}_A\Delta T, \tag{12}$$

$$\hat{\varepsilon}_T = -\frac{\hat{\eta}_t}{\hat{E}_T}\sigma_t - \frac{\hat{\eta}_A}{\hat{E}_T}\sigma - \frac{\hat{\eta}_T}{\hat{E}_T}\sigma_T - \frac{\hat{\delta}_A}{\hat{E}_A}M + \frac{M_T}{\hat{E}_T} + \hat{\alpha}_T\Delta T, \tag{13}$$

which defines the various thermo-elastic constants that characterise the properties of a damaged cross-ply laminate subject to combined in-plane biaxial loading, out-of-plane through-thickness loading and biaxial bending. In (9-13), ΔT is the temperature difference defined by $\Delta T = T - T_0$, where T is the current temperature of the material, and T_0 is the 'manufacturing' temperature at which the strain is zero and the material is everywhere stress-free, with no internal or imposed external stresses and displacements.

The stress-strain-temperature relations of the i^{th} ply in the laminate are assumed to be of the following linear orthotropic form

$$\varepsilon_t^i = \frac{\sigma_t}{E_t^i} - \frac{\nu_a^i}{E_A^i}\sigma^i - \frac{\nu_t^i}{E_T^i}\sigma_T^i + \alpha_t^i\Delta T, \tag{14}$$

$$\varepsilon^i = -\frac{\nu_a^i}{E_A^i}\sigma_t + \frac{\sigma^i}{E_A^i} - \frac{\nu_A^i}{E_A^i}\sigma_T^i + \alpha_A^i\Delta T, \tag{15}$$

$$\varepsilon_T^i = -\frac{\nu_t}{E_T}\sigma_t - \frac{\nu_A}{E_A}\sigma + \frac{\sigma_T}{E_T} + \alpha_T\Delta T, \tag{16}$$

where ε_t^i, ε^i and ε_T^i are the through-thickness, axial and transverse in-plane strains. The in-plane axial and transverse stresses in the plies are denoted by σ^i and σ_T^i while the through-thickness stress has the same value σ_t in all the plies of the laminate, and in the homogenised equivalent laminate. The parameters E, ν and α denote the Young's modulus, Poisson's ratio and thermal expansion coefficient respectively. The thermo-elastic constants E, ν and α are allowed to have different values in each of the plies of the equivalent laminate so that the analysis can be applied to hybrid laminates where the plies of a multiple-ply laminate are made of different materials. The upper case subscripts A and T are attached to axial and transverse thermo-elastic constants to denote that they refer to in-plane stresses and deformation while the corresponding lower case subscripts denote thermo-elastic constants that involve out-of-plane stresses and deformations. Each ply has been assumed to be orthotropic so that twelve thermo-elastic constants are required to characterise linear behaviour when the three shear moduli are included (not shown above). It should be noted that the form of the stress/strain relations (14-16) has assumed that the axial direction of each ply is oriented in the direction of the fibres in the 0^o plies of the laminate. If the 0^o and 90^o plies of the laminate are made of the same material, then for the 0^o plies

$$\mathrm{E}_{\mathrm{t}}^{0} = E_t \ , \quad \mathrm{E}_{\mathrm{A}}^{0} = E_A \ , \quad \mathrm{E}_{\mathrm{T}}^{0} = E_T \ , \quad \nu_{\mathrm{t}}^{0} = \nu_t \ , \quad \nu_{\mathrm{A}}^{0} = \nu_A \ , \quad \nu_{\mathrm{a}}^{0} = \nu_a \ ,$$
$$\alpha_{\mathrm{t}}^{0} = \alpha_t \ , \quad \alpha_{\mathrm{A}}^{0} = \alpha_A \ , \quad \alpha_{\mathrm{T}}^{0} = \alpha_T \ ,$$

where E_t, E_A, E_T, ν_t, ν_A, ν_a, α_t, α_A and α_T are the thermo-elastic constants of the plies of the composite that can be measured values. For the 90^o plies the following identifications must be made

$$\mathrm{E}_{\mathrm{t}}^{90} = E_t \ , \quad \mathrm{E}_{\mathrm{A}}^{90} = E_T \ , \quad \mathrm{E}_{\mathrm{T}}^{90} = E_A \ , \quad \nu_{\mathrm{t}}^{90} = \nu_{\mathrm{a}} \ , \quad \nu_{\mathrm{A}}^{90} = \nu_A E_T / E_A \ , \quad \nu_{\mathrm{a}}^{90} = \nu_T \ ,$$
$$\alpha_{\mathrm{t}}^{90} = \alpha_t \ , \quad \alpha_{\mathrm{A}}^{90} = \alpha_T \ , \quad \alpha_{\mathrm{T}}^{90} = \alpha_A \ ,$$

The above approach has been used to investigate ply crack formation in cross-ply laminates [2, 3, 6-8]. A methodology for extending this approach to general balanced laminates will now be described. The laminate must be balanced so that twisting deformations are avoided, but it can be unsymmetric.

2.3 An homogenisation methodology

As detailed solutions are available only for the case of multiple-ply *cross-ply* laminates subject to combined in-plane biaxial loading and out-of-plane bending, it is necessary to develop an homogenisation technique for undamaged laminates that will replace balanced off-axis plies in a general laminate by an equivalent single 0^o ply having properties that match the effective properties of the off-axis plies that have been replaced using the homogenisation technique. The stress-strain relations for an off-axis ply in terms of global coordinates, where the axial direction corresponds with fibre direction of the 0^o plies in the laminate, are given by

$$\varepsilon_{\mathrm{xx}}^{\mathrm{i}} = \mathrm{g}_{11}^{\mathrm{i}}\sigma_{\mathrm{xx}}^{\mathrm{i}} + \mathrm{g}_{12}^{\mathrm{i}}\sigma_{\mathrm{yy}}^{\mathrm{i}} + \mathrm{g}_{13}^{\mathrm{i}}\sigma_{\mathrm{zz}}^{\mathrm{i}} + \mathrm{g}_{14}^{\mathrm{i}}\sigma_{\mathrm{yz}}^{\mathrm{i}} + \alpha_1^{\mathrm{i}}\Delta\mathrm{T} \ , \tag{17}$$

$$\varepsilon_{\mathrm{yy}}^{\mathrm{i}} = \mathrm{g}_{12}^{\mathrm{i}}\sigma_{\mathrm{xx}}^{\mathrm{i}} + \mathrm{g}_{22}^{\mathrm{i}}\sigma_{\mathrm{yy}}^{\mathrm{i}} + \mathrm{g}_{23}^{\mathrm{i}}\sigma_{\mathrm{zz}}^{\mathrm{i}} + \mathrm{g}_{24}^{\mathrm{i}}\sigma_{\mathrm{yz}}^{\mathrm{i}} + \alpha_2^{\mathrm{i}}\Delta\mathrm{T} \ , \tag{18}$$

$$\varepsilon_{\mathrm{zz}}^{\mathrm{i}} = \mathrm{g}_{13}^{\mathrm{i}}\sigma_{\mathrm{xx}}^{\mathrm{i}} + \mathrm{g}_{23}^{\mathrm{i}}\sigma_{\mathrm{yy}}^{\mathrm{i}} + \mathrm{g}_{33}^{\mathrm{i}}\sigma_{\mathrm{zz}}^{\mathrm{i}} + \mathrm{g}_{34}^{\mathrm{i}}\sigma_{\mathrm{yz}}^{\mathrm{i}} + \alpha_3^{\mathrm{i}}\Delta\mathrm{T} \ , \tag{19}$$

$$2\varepsilon_{\mathrm{yz}}^{\mathrm{i}} = \mathrm{g}_{14}^{\mathrm{i}}\sigma_{\mathrm{xx}}^{\mathrm{i}} + \mathrm{g}_{24}^{\mathrm{i}}\sigma_{\mathrm{yy}}^{\mathrm{i}} + \mathrm{g}_{34}^{\mathrm{i}}\sigma_{\mathrm{zz}}^{\mathrm{i}} + \mathrm{g}_{44}^{\mathrm{i}}\sigma_{\mathrm{yz}}^{\mathrm{i}} + \alpha_4^{\mathrm{i}}\Delta\mathrm{T} \ , \tag{20}$$

$$\varepsilon_{\mathrm{xy}}^{\mathrm{i}} = \mathrm{a}_{11}^{\mathrm{i}}\sigma_{\mathrm{xy}}^{\mathrm{i}} + \mathrm{a}_{12}^{\mathrm{i}}\sigma_{\mathrm{xz}}^{\mathrm{i}} \ , \tag{21}$$

$$\varepsilon_{\mathrm{xz}}^{\mathrm{i}} = \mathrm{a}_{12}^{\mathrm{i}}\sigma_{\mathrm{xy}}^{\mathrm{i}} + \mathrm{a}_{22}^{\mathrm{i}}\sigma_{\mathrm{xz}}^{\mathrm{i}} \ . \tag{22}$$

The various coefficients appearing in (17-22) are defined, in terms of ply properties referred to local coordinates, in reference [8]. It is shown in this reference how the effective properties of any symmetric laminate may be found, which are defined by the following global stress-strain relations

$$\varepsilon_{\mathrm{t}} = \frac{\sigma_{\mathrm{t}}}{\overline{\mathrm{E}}_{\mathrm{t}}} - \frac{\overline{\nu}_{\mathrm{a}}}{\overline{\mathrm{E}}_{\mathrm{A}}}\sigma - \frac{\overline{\nu}_{\mathrm{t}}}{\overline{\mathrm{E}}_{\mathrm{T}}}\sigma_{\mathrm{T}} + \alpha_{\mathrm{t}}\Delta\mathrm{T} \ , \tag{23}$$

$$\varepsilon = -\frac{\overline{\nu}_{\mathrm{a}}}{\overline{\mathrm{E}}_{\mathrm{A}}}\sigma_{\mathrm{t}} + \frac{\sigma}{\overline{\mathrm{E}}_{\mathrm{A}}} - \frac{\overline{\nu}_{\mathrm{A}}}{\overline{\mathrm{E}}_{\mathrm{A}}}\sigma_{\mathrm{T}} + \overline{\alpha}_{\mathrm{A}}\Delta\mathrm{T} \ , \tag{24}$$

$$\varepsilon_{\mathrm{T}} = -\frac{\overline{\nu}_{\mathrm{t}}}{\overline{\mathrm{E}}_{\mathrm{T}}}\sigma_{\mathrm{t}} - \frac{\overline{\nu}_{\mathrm{A}}}{\overline{\mathrm{E}}_{\mathrm{A}}}\sigma + \frac{\sigma_{\mathrm{T}}}{\overline{\mathrm{E}}_{\mathrm{T}}} + \overline{\alpha}_{\mathrm{T}}\Delta\mathrm{T} \ . \tag{25}$$

The stress-strain relations (23-25), where the over-bar denotes the homogenised properties, are identical in form to those given by (14-16). It is thus entirely feasible to replace undamaged off-axis plies in a general laminate by 0^{o} plies having equivalent properties specified in (23-25). It is emphasised, however, that this is not the case when some of the plies might be damaged due to ply cracking.

2.4 Reduced stress-strain relations for constrained triaxial loading

On solving (12) and (13) for M and M_T and substituting into the remaining damage-dependent stress-strain relations (9-11), the following reduced stress-strain relations are derived of the same form as those for a cross-ply laminate subject only to triaxial loading, without shear or bending

$$\tilde{\varepsilon}_t = \varepsilon_t + \frac{\hat{\nu}_a}{\Lambda}\left[\hat{\varepsilon} + \hat{\delta}_T\hat{\varepsilon}_T\right] + \frac{\hat{\eta}_t}{\Lambda}\left[\hat{\varepsilon}_T + \hat{\delta}_A\hat{\varepsilon}\right] = \frac{\sigma_t}{\tilde{E}_t} - \frac{\tilde{\nu}_a}{\tilde{E}_A}\sigma - \frac{\tilde{\nu}_t}{\tilde{E}_T}\sigma_T + \tilde{\alpha}_t\Delta T , \tag{26}$$

$$\tilde{\varepsilon} = \hat{\varepsilon} + \frac{\hat{\nu}_A}{\Lambda}\left[\hat{\varepsilon} + \hat{\delta}_T\hat{\varepsilon}_T\right] + \frac{\hat{\eta}_A}{\Lambda}\left[\hat{\varepsilon}_T + \hat{\delta}_A\hat{\varepsilon}\right] = -\frac{\tilde{\nu}_a}{\tilde{E}_A}\sigma_t + \frac{\sigma}{\tilde{E}_A} - \frac{\tilde{\nu}_A}{\tilde{E}_A}\sigma_T + \tilde{\alpha}_A\Delta T , \tag{27}$$

$$\tilde{\varepsilon}_T = \hat{\varepsilon}_T + \frac{\hat{\nu}_T}{\Lambda}\left[\hat{\varepsilon} + \hat{\delta}_T\hat{\varepsilon}_T\right] + \frac{\hat{\eta}_T}{\Lambda}\left[\hat{\varepsilon}_T + \hat{\delta}_A\hat{\varepsilon}\right] = -\frac{\tilde{\nu}_t}{\tilde{E}_T}\sigma_t - \frac{\tilde{\nu}_A}{\tilde{E}_A}\sigma + \frac{\sigma_T}{\tilde{E}_T} + \tilde{\alpha}_T\Delta T , \tag{28}$$

with
$$\hat{\delta}_T = \hat{\delta}_A\frac{\hat{E}_T}{\hat{E}_A}, \qquad \Lambda = 1 - \hat{\delta}_A\hat{\delta}_T , \tag{29}$$

where the reduced strains $\tilde{\varepsilon}_t$, $\tilde{\varepsilon}$ and $\tilde{\varepsilon}_T$ can be interpreted as strains for a damaged laminate, subject to triaxial loading and constrained so that bending strains are zero, and where the reduced thermoelastic constants are defined by

$$\frac{1}{\tilde{E}_t} = \frac{1}{E_t} - \frac{1}{\Lambda}\left[\frac{\hat{\nu}_a^2}{\hat{E}_A} + \frac{\hat{\eta}_t^2}{\hat{E}_T} + \frac{2\hat{\delta}_A\hat{\nu}_a\hat{\eta}_t}{\hat{E}_A}\right], \qquad \frac{1}{\tilde{E}_A} = \frac{1}{E_A} - \frac{1}{\Lambda}\left[\frac{\hat{\nu}_A^2}{\hat{E}_A} + \frac{\hat{\eta}_A^2}{\hat{E}_T} + \frac{2\hat{\delta}_A\hat{\nu}_A\hat{\eta}_A}{\hat{E}_A}\right],$$

$$\frac{1}{\tilde{E}_T} = \frac{1}{E_T} - \frac{1}{\Lambda}\left[\frac{\hat{\nu}_T^2}{\hat{E}_A} + \frac{\hat{\eta}_T^2}{\hat{E}_T} + \frac{2\hat{\delta}_A\hat{\nu}_T\hat{\eta}_T}{\hat{E}_A}\right],$$

$$\frac{\tilde{\nu}_a}{\tilde{E}_A} = \frac{\nu_a}{E_A} + \frac{1}{\Lambda}\left[\frac{\hat{\nu}_a}{\hat{E}_A}\left(\hat{\nu}_A + \hat{\delta}_A\hat{\eta}_A\right) + \frac{\hat{\eta}_t}{\hat{E}_T}\left(\hat{\delta}_T\hat{\nu}_A + \hat{\eta}_A\right)\right],$$

$$\frac{\tilde{\nu}_t}{\tilde{E}_T} = \frac{\nu_t}{E_T} + \frac{1}{\Lambda}\left[\frac{\hat{\nu}_a}{\hat{E}_A}\left(\hat{\nu}_T + \hat{\delta}_A\hat{\eta}_T\right) + \frac{\hat{\eta}_t}{\hat{E}_T}\left(\hat{\delta}_T\hat{\nu}_T + \hat{\eta}_T\right)\right],$$

$$\frac{\tilde{\nu}_A}{\tilde{E}_A} = \frac{\nu_A}{E_A} + \frac{1}{\Lambda}\left[\frac{\hat{\nu}_A}{\hat{E}_A}\left(\hat{\nu}_T + \hat{\delta}_A\hat{\eta}_T\right) + \frac{\hat{\eta}_A}{\hat{E}_T}\left(\hat{\delta}_T\hat{\nu}_T + \hat{\eta}_T\right)\right],$$

$$\tilde{\alpha}_t = \alpha_t + \frac{1}{\Lambda}\left[\hat{\nu}_a\left(\hat{\delta}_T\hat{\alpha}_T + \hat{\alpha}_A\right) + \hat{\eta}_t\left(\hat{\alpha}_T + \hat{\delta}_A\hat{\alpha}_A\right)\right],$$

$$\tilde{\alpha}_A = \alpha_A + \frac{1}{\Lambda}\left[\hat{\nu}_A\left(\hat{\delta}_T\hat{\alpha}_T + \hat{\alpha}_A\right) + \hat{\eta}_A\left(\hat{\alpha}_T + \hat{\delta}_A\hat{\alpha}_A\right)\right],$$

$$\tilde{\alpha}_T = \alpha_T + \frac{1}{\Lambda}\left[\hat{\nu}_T\left(\hat{\delta}_T\hat{\alpha}_T + \hat{\alpha}_A\right) + \hat{\eta}_T\left(\hat{\alpha}_T + \hat{\delta}_A\hat{\alpha}_A\right)\right].$$

The corresponding reduced stress-strain relations for undamaged laminates are written:

$$\tilde{\varepsilon}_t^{o} = \frac{\sigma_t}{\tilde{E}_t^{o}} - \frac{\tilde{\nu}_a^{o}}{\tilde{E}_A^{o}}\sigma - \frac{\tilde{\nu}_t^{o}}{\tilde{E}_T^{o}}\sigma_T + \tilde{\alpha}_t^{o}\Delta T , \tag{30}$$

$$\tilde{\varepsilon}^{\mathrm{o}} = -\frac{\tilde{\nu}_{\mathrm{a}}^{\mathrm{o}}}{\tilde{E}_{\mathrm{A}}^{\mathrm{o}}}\sigma_{\mathrm{t}} + \frac{\sigma}{\tilde{E}_{\mathrm{A}}^{\mathrm{o}}} - \frac{\tilde{\nu}_{\mathrm{A}}^{\mathrm{o}}}{\tilde{E}_{\mathrm{A}}^{\mathrm{o}}}\sigma_{\mathrm{T}} + \tilde{\alpha}_{\mathrm{A}}^{\mathrm{o}}\Delta T \ , \tag{31}$$

$$\tilde{\varepsilon}_{\mathrm{T}}^{\mathrm{o}} = -\frac{\tilde{\nu}_{\mathrm{t}}^{\mathrm{o}}}{\tilde{E}_{\mathrm{T}}^{\mathrm{o}}}\sigma_{\mathrm{t}} - \frac{\tilde{\nu}_{\mathrm{A}}^{\mathrm{o}}}{\tilde{E}_{\mathrm{A}}^{\mathrm{o}}}\sigma + \frac{\sigma_{\mathrm{T}}}{\tilde{E}_{\mathrm{T}}^{\mathrm{o}}} + \tilde{\alpha}_{\mathrm{T}}^{\mathrm{o}}\Delta T \ , \tag{32}$$

where a superscript ‘o’ denotes that the strains and laminate properties refer to their values for the undamaged state of the laminate.

2.5 Fundamental inter-relationships between thermo-elastic constants

By considering the conditions for ply crack closure during uniaxial loading in the axial, transverse and through-thickness directions, it can be shown [8] that many inter-relationships between the thermoelastic constants of a damaged laminate can be derived. First of all define the damage parameter

$$\tilde{D} = \frac{1}{\tilde{E}_{\mathrm{A}}} - \frac{1}{\tilde{E}_{\mathrm{A}}^{\mathrm{o}}} \ . \tag{33}$$

It has been shown [8] that the thermo-elastic constants for a damaged laminate are related to those of the corresponding undamaged laminate according to the following simple relations

$$\frac{1}{\tilde{E}_{\mathrm{t}}} - \frac{1}{\tilde{E}_{\mathrm{t}}^{\mathrm{o}}} = \left(\tilde{k}'\right)^2 \tilde{D} \ , \qquad \frac{1}{\tilde{E}_{\mathrm{A}}} - \frac{1}{\tilde{E}_{\mathrm{A}}^{\mathrm{o}}} = \tilde{D} \ , \qquad \frac{1}{\tilde{E}_{\mathrm{T}}} - \frac{1}{\tilde{E}_{\mathrm{T}}^{\mathrm{o}}} = \tilde{k}^2 \tilde{D} \ , \tag{34}$$

$$\frac{\tilde{\nu}_{\mathrm{t}}^{\mathrm{o}}}{\tilde{E}_{\mathrm{T}}^{\mathrm{o}}} - \frac{\tilde{\nu}_{\mathrm{t}}}{\tilde{E}_{\mathrm{T}}} = \tilde{k}\,\tilde{k}'\tilde{D} \ , \qquad \frac{\tilde{\nu}_{\mathrm{A}}^{\mathrm{o}}}{\tilde{E}_{\mathrm{A}}^{\mathrm{o}}} - \frac{\tilde{\nu}_{\mathrm{A}}}{\tilde{E}_{\mathrm{A}}} = \tilde{k}\,\tilde{D} \ , \qquad \frac{\tilde{\nu}_{\mathrm{a}}^{\mathrm{o}}}{\tilde{E}_{\mathrm{A}}^{\mathrm{o}}} - \frac{\tilde{\nu}_{\mathrm{a}}}{\tilde{E}_{\mathrm{A}}} = \tilde{k}'\tilde{D} \ , \tag{35}$$

$$\tilde{\alpha}_{\mathrm{t}} - \tilde{\alpha}_{\mathrm{t}}^{\mathrm{o}} = \tilde{k}'\tilde{k}_1\tilde{D} \ , \qquad \tilde{\alpha}_{\mathrm{A}} - \tilde{\alpha}_{\mathrm{A}}^{\mathrm{o}} = \tilde{k}_1\tilde{D} \ , \qquad \tilde{\alpha}_{\mathrm{T}} - \tilde{\alpha}_{\mathrm{T}}^{\mathrm{o}} = \tilde{k}\,\tilde{k}_1\tilde{D} \ . \tag{36}$$

The undamaged laminate constants $\tilde{k}_1$, $\tilde{k}$ and $\tilde{k}'$ are defined by

$$\left.\begin{aligned} \tilde{k}_1 &= \frac{\tilde{E}_{\mathrm{A}}^{\mathrm{o}}\left[\tilde{\alpha}_{\mathrm{A}}^{\mathrm{o}} + B\,\tilde{\alpha}_{\mathrm{T}}^{\mathrm{o}} - C\right]}{1 - \tilde{\nu}_{\mathrm{A}}^{\mathrm{o}} B} \ , \qquad \tilde{k} = \frac{\tilde{E}_{\mathrm{A}}^{\mathrm{o}}}{\tilde{E}_{\mathrm{T}}^{\mathrm{o}}}\,\frac{B - \tilde{\nu}_{\mathrm{A}}^{\mathrm{o}}\tilde{E}_{\mathrm{T}}^{\mathrm{o}} / \tilde{E}_{\mathrm{A}}^{\mathrm{o}}}{1 - \tilde{\nu}_{\mathrm{A}}^{\mathrm{o}} B} \ , \\ \tilde{k}' &= \frac{\tilde{E}_{\mathrm{A}}^{\mathrm{o}} A - \tilde{\nu}_{\mathrm{a}}^{\mathrm{o}} - \tilde{\nu}_{\mathrm{t}}^{\mathrm{o}}\tilde{E}_{\mathrm{A}}^{\mathrm{o}} / \tilde{E}_{\mathrm{T}}^{\mathrm{o}} B}{1 - \tilde{\nu}_{\mathrm{A}}^{\mathrm{o}} B} \ , \end{aligned}\right\} \tag{37}$$

where the parameters A, B and C are laminate constants defined in reference [8, eq.(39)].

2.6 Gibbs free energy for a cracked laminate subject to multi-axial bending

It has been shown [7], for the case of uniform ply crack densities in one or more of the 90$^{\mathrm{o}}$ plies of the laminate, that the Gibbs free energy (equivalent to the complementary energy) per unit volume of laminate (averaged over a region V occupied by the laminate) may be expressed in the form

$$\langle g\rangle - \langle g_0\rangle = -\tfrac{1}{2}\tilde{D}\left[\, s - \sigma_{\mathrm{c}}\right]^2 - F(\hat{\varepsilon}, \hat{\varepsilon}_{\mathrm{T}}) + F_0(\hat{\varepsilon}^{\mathrm{o}}, \hat{\varepsilon}_{\mathrm{T}}^{\mathrm{o}}) \ , \tag{38}$$

where $\langle g_0\rangle$ is the value of $\langle g\rangle$ for an undamaged laminate, where $s = \tilde{k}'\sigma_{\mathrm{t}} + \sigma + \tilde{k}\,\sigma_{\mathrm{T}}$ is an effective stress, and where σ_{c} is the crack closure stress for uniaxial in-plane loading constrained so that there is no bending, and where

$$\left.\begin{aligned} F(\hat{\varepsilon}, \hat{\varepsilon}_{\mathrm{T}}) &= \frac{1}{2\Lambda}\left[\,\hat{E}_{\mathrm{A}}\hat{\varepsilon}\left(\hat{\varepsilon} + \hat{\delta}_{\mathrm{T}}\hat{\varepsilon}_{\mathrm{T}}\right) + \hat{E}_{\mathrm{T}}\hat{\varepsilon}_{\mathrm{T}}\left(\hat{\varepsilon}_{\mathrm{T}} + \hat{\delta}_{\mathrm{A}}\hat{\varepsilon}\right)\right], \qquad \Lambda = 1 - \hat{\delta}_{\mathrm{A}}\hat{\delta}_{\mathrm{T}} \ , \\ F_0(\hat{\varepsilon}^{\mathrm{o}}, \hat{\varepsilon}_{\mathrm{T}}^{\mathrm{o}}) &= \frac{1}{2\Lambda_0}\left[\,\hat{E}_{\mathrm{A}}^{\mathrm{o}}\hat{\varepsilon}^{\mathrm{o}}\left(\hat{\varepsilon}^{\mathrm{o}} + \hat{\delta}_{\mathrm{T}}^{\mathrm{o}}\hat{\varepsilon}_{\mathrm{T}}^{\mathrm{o}}\right) + \hat{E}_{\mathrm{T}}^{\mathrm{o}}\hat{\varepsilon}_{\mathrm{T}}^{\mathrm{o}}\left(\hat{\varepsilon}_{\mathrm{T}}^{\mathrm{o}} + \hat{\delta}_{\mathrm{A}}^{\mathrm{o}}\hat{\varepsilon}^{\mathrm{o}}\right)\right], \qquad \Lambda_0 = 1 - \hat{\delta}_{\mathrm{A}}^{\mathrm{o}}\hat{\delta}_{\mathrm{T}}^{\mathrm{o}} \ . \end{aligned}\right\} \tag{39}$$

2.7 Predicting damage formation

Consider the special case where, during the formation of every ply crack, the fracture energy for ply crack formation has a unique value 2γ. The first objective is to determine the conditions for which it is energetically favourable for an array of equally spaced ply cracks having density ρ_0 to form quasi-statically in an undamaged laminate subject to fixed applied loads and temperature. For a macroscopic region V of the laminate, energy balance considerations and the fact that kinetic energy is never negative, lead to the following criterion for crack formation, involving the change of Gibbs free energy ΔG (equivalent to complementary free energy) having the form

$$\Delta\Gamma \;+\; \Delta G \;<\; 0\,, \tag{40}$$

where the energy absorbed in a macroscopic volume V of laminate by the formation of the new ply cracks is given by

$$\Delta\Gamma \;=\; V\Gamma \;=\; V\,\frac{2\gamma\,\rho_0\,h^{(90)}}{h}\,. \tag{41}$$

In (40), the parameter Γ denotes the energy absorbed per unit volume of laminate during the formation of new ply crack surfaces in the 90^o plies that have led to the initial damage state denoted by the ply crack density ρ_0, and $2h^{(90)}$ is the total thickness of the 90^o plies in which the ply cracks have formed. It then follows that the first ply cracking stress s^1 may be determined from the following inequality

$$\frac{1}{2}D(\rho_0)\left(s^1-\sigma^c\right)^2 + F(\hat{\varepsilon}^1,\hat{\varepsilon}_T^1,\rho_0) - F_0(\hat{\varepsilon}^0,\hat{\varepsilon}_T^0) > \frac{2\gamma\rho_0 h^{(90)}}{h}\,. \tag{42}$$

Subsequent progressive ply crack formation is predicted by applying successively, for values i = 1, 2, … , the inequality

$$\frac{1}{2}\left[D(\rho_{i+1})-D(\rho_i)\right]\left(s^{i+1}-\sigma^c\right)^2 + F(\hat{\varepsilon}^{i+1},\hat{\varepsilon}_T^{i+1},\rho_{i+1}) - F(\hat{\varepsilon}^{i},\hat{\varepsilon}_T^{i},\rho_{i}) > \frac{2^{i+1}\gamma\rho_0 h^{(90)}}{h}\,. \tag{43}$$

It has been shown [8] that the approach described above does not provide any information that indicates how the thermo-elastic constants depend upon ply crack density. A detailed stress analysis is required to provide this information (see [6]).

3. Example predictions

A quasi-isotropic $[45/-45/0/90]_s$ laminate is considered, having ply thickness d, that is to be subjected to combined in-plane biaxial loading and out-of-plane biaxial bending in the presence of thermal residual stresses, where anticlastic bending is taken into account. One half of the symmetric laminate considered is shown in Figure 1 together with two options for homogenising the plies that do not have a 90^o orientation. For Model 1 shown in Figure 1, the $\pm 45^o$ plies on each side of the laminate are replaced by homogenised plies of thickness 2d whose stress-strain relations will be of the form (23-25). For Model 2 shown in Figure 1, the 0^o and $\pm 45^o$ plies on each side of the laminate are homogenised into plies of thickness 3d. The configurations shown in Figure 1 for Models 1 and 2 can be analysed using the methods described in Section 2.

A key issue is how well the new model behaves when compared with other solution techniques for the special case when there are no bending loads. The selected test subjects the models, for typical CFRP and GRP laminates, to uniaxial loading without bending or shear, for the case when the stress-free temperature has the value $\Delta T = -85^o C$. Figures 2 and 3 show a comparison of results for CFRP and GRP laminates, respectively.

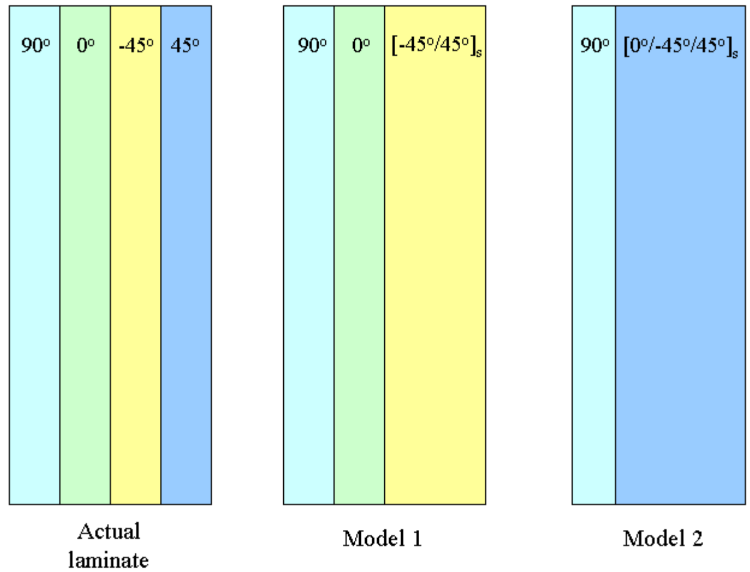

Figure 1: Diagram illustrating two possible homogenised models of a laminate.

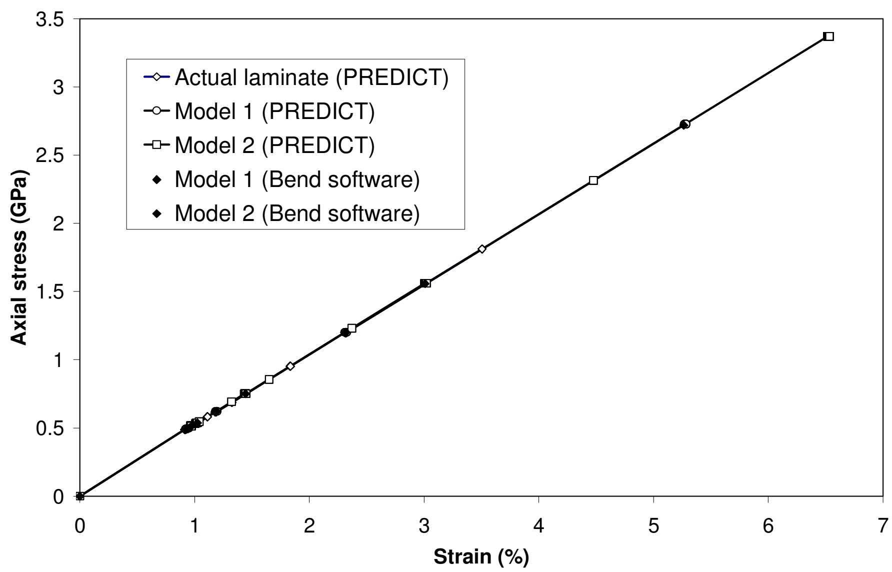

Figure 2: Comparison of predictions for a typical CFRP laminate using various models.

Table 1: First ply cracking for the CFRP laminate.

Model	ρ_0 (/mm)	ε (%)	σ (GPa)
Actual	0.729904	0.9218	0.4879
Model 1	0.729904	0.92945	0.4920
Model 2	0.941764	0.9713	0.51325
Model 1 (Bend)	0.729905	0.9297	0.4932
Model 2 (Bend)	0.941762	0.9774	0.5161

The software system PREDICT [9], based on analysis given in reference [8] and associated papers and assuming a zero standard deviation for the fracture energies having a precise mean value taken as 150 J.m^{-2} for both CFRP and GRP laminates, has been used to generate the results in Figures 2 and 3 that are shown as continuous lines with open symbols. The new model for bending deformation developed in this paper has been used to determine the results shown as closed symbols. Clearly there is extremely good agreement between the stress-strain curves predicted, for both CFRP and GRP laminates, by the various models, validating the new homogenisation model for the special case when bending deformation is absent. There are, however, some differences that can be more easily identified by considering the axial first ply cracking stresses σ, axial strains ε and initial crack densities ρ_0, which are given in Tables 1 and 2. The ply cracking stresses and strains are highly sensitive to the ply crack density and the six decimal places given in Tables 1 and 2 are in fact significant.

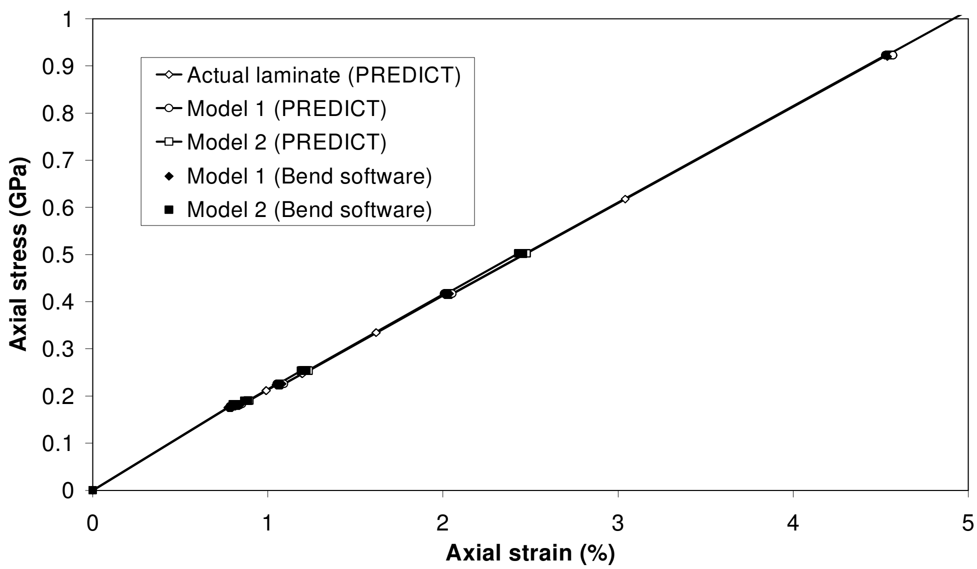

Figure 3: Comparison of predictions for a typical GRP laminate using various models.

Table 2: First ply cracking for the GRP laminate.

Model	ρ_0 (/mm)	ε (%)	σ (GPa)
Actual	0.693854	0.7974	0.1755
Model 1	0.693854	0.8080	0.17795
Model 2	0.827335	0.8253	0.1809
Model 1 (Bend)	0.693853	0.8062	0.1784
Model 2 (Bend)	0.827335	0.8249	0.1818

It is seen that when the 0^o plies are included in the homogenisation procedure (as for Model 2), the initial ply crack density is significantly different to that for the other cases. For the homogenisation leading to Model 1, the results are very similar to those obtained for the actual laminate where homogenisation has not been used. It is deduced that any homogenised approach should handle discretely the 0^o plies in a laminate. It remains now to provide some results for an example illustration of progressive ply cracking in a CFRP quasi-isotropic laminate that is subject to complex proportional loading such that

$$\sigma_t = 0,\ \sigma = S,\ \sigma_T = 0.2S,\ M = 0.3S,\ M_T = 0.1S,\ \Delta T = -85^o C,\ 2\gamma = 150\,\mathrm{J.m^{-2}},$$

where applied stresses are measured in GPa (i.e. kN.mm^{-2}), and where applied bending moments per unit cross-sectional area are measured in kN.mm^{-1}. Table 3 shows the results obtained, for the axial stresses σ, axial strain ε and the bending curvatures $\hat{\varepsilon}$

and $\hat{\varepsilon}_T$, using Model 1 of the new homogenisation model, which are such that ply crack densities ρ at each successive stage progressively double.

Table 3: Progressive ply cracking under complex loading in a CFRP laminate.

ρ (/mm)	σ (GPa)	ε (%)	$\hat{\varepsilon}$ (%) (/mm)	$\hat{\varepsilon}_T$ (%) (/mm)
0.728654	0.490	0.868	4.276	-1.705
1.457308	0.499	0.890	4.353	-1.736
2.914616	0.612	1.104	5.346	-2.133
5.829232	1.138	2.064	9.938	-3.966

Work is in progress to extend the model so that a random distribution of fracture energies can be considered (as in PREDICT [9]), leading to progressive ply crack formation where cracks form one at a time.

4. Conclusion

On the basis of the limited results presented in this paper, it is concluded that the proposed homogenisation method (using Model 1) is a good basis for predicting ply crack formation in general laminates subject to complex loading involving bend deformation, in addition to in-plane biaxial loading and thermal residual stresses.

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   (see http://www.npl.co.uk/npl/cmmt/cog/coda.html)