ANALYTICAL METHODS FOR THE STRESS CONCENTRATION ANALYSIS OF MULTILAYERED ANISOTROPIC COMPOSITES WITH PIN-LOADED HOLES

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ABSTRACT

The use of fibre- and textile-reinforced multilayered composites in particular for load-carrying and safety-relevant applications has been expanded considerably during recent years. Especially textile semi-finished reinforcing products in form of multi-axial knitted, woven or braided performs have gained increasing importance compared to uni- or bidirectional fibre-reinforced layers.
For these kinds of materials the problem of stress concentrations in the vicinity of pin-loaded holes is of great importance for the design of the whole composite structure. To pre-calculate the stress, strain and displacement fields in such problem zones, analytical methods offer decisive advantages since they, in contrast to numerical methods, allow efficient weighting of the influence of the single parameters and in this way permit a physical interpretation of complicated notch phenomena. Therefore advanced analytical solution methods for the stress concentration analysis of multilayered composites with pin-loaded holes were developed at the Institut für Leichtbau und Kunststofftechnik (ILK) on the basis of layer-related solutions and have been confirmed in numerical calculations.

1. INTRODUCTION

The application of fibre- or textile-reinforced composite materials has been expanded considerably in recent years. While up to now the reinforcing structure mostly was constructed of uni- or bidirectional fibre-reinforced laminae, nowadays semi-finished textile products in form of multi-axial knitted, woven or braided preforms are getting more and more in the focus of research and application. For composite structures, the analysis of the stress concentration behaviour in the vicinity of cut-outs, elastic inclusions or pin-loaded holes is of great importance since holes in composites can often be considered as design drivers for the whole structure. Related issues occur as subproblems during the evaluation of rivets, screws, etc.
Whilst in literature analytical solutions for the stress concentration problem of fibre-reinforced multilayered composite plates with cut-outs and elastic inclusions can be found [1-2] (mostly based on the fundamental works of LEKHNITSKII [3]), such publications are hardly found for generally multilayered anisotropic composites with pin-loaded holes [4-6].
During recent years, sophisticated analytical solutions for basic problems concerning the stress concentration analysis of notched layered composites have already been developed at the Institut für Leichtbau und Kunststofftechnik (ILK) [5-9]. Based on these experiences, an analytical method was developed, which enables a layer-by-layer pre-calculation of the entire stress and distortion field for generally structured multilayered composites with pin-loaded holes.
For this purpose, the model of an infinite plate with a sinusoidal distribution of normal edge forces on a continuous part of the notch edge was chosen as a mechanical equivalent (partial edge load) for the pin-loaded hole. Taking into consideration the extension-bending coupling effects occurring in asymmetrical composites and thus not

to have any restrictions with regard to the composite lay-up, it is necessary to pursue superordinate approaches in the expanded stress-deformation analysis of generally anisotropic multilayered composites (MLC).

2. ANALYTICAL CALCULATION METHOD

2.1. Plate equation

The starting point for the presented calculation method is the expanded structural law for multilayered composites as known from the classical laminate theory (CLT) taking into consideration the extension-bending coupling effects occurring in asymmetrical composites (1)

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x^0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{bmatrix}, \tag{1}$$

with

N_i, M_i	as force resultants and moment resultants ($i = x, y, xy$),
(A_{ij}), (B_{ij}), (D_{ij})	as extensional, extension-bending coupling and bending stiffnesses of multilayered composites ($i, j = 1, 2, 6$),
ε_i^0, γ_j^0, κ_k^0	as distortions of the neutral plane ($i = x, y; j = xy; k = x, y, xy$).

Due to the extension-bending coupling effects that can occur in generally multilayered fibre-reinforced composites, a separation of the plate-bending and membrane problems often is not possible. Therefore, starting from the classic plate theory by KIRCHHOFF and supplementing the equilibrium of force and moment resultants by the membrane force resultants, a generalized plate equation is derived, which in particular considers the coupling effects in asymmetric composites (see also [8]):

$$\underline{\underline{\Delta}} \begin{bmatrix} (A_{ij}) & (B_{ij}) \\ (B_{ij}) & (D_{ij}) \end{bmatrix} \underline{\underline{\Delta}}^T \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = -\underline{\underline{\Delta}} \begin{bmatrix} (N_i) \\ (M_i) \end{bmatrix} \tag{2}$$

with the differential operator matrix

$$\underline{\underline{\Delta}}^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial y^2} & -2\frac{\partial^2}{\partial x \partial y} \end{bmatrix} \tag{3}$$

and

u_0, v_0, w_0 as the displacements of the neutral plane.

2.2. Complex-valued displacement functions and conformal mapping

An infinite plate with a circular or elliptical cut-out is selected as mathematically equivalent model for the purpose of dealing with the stress concentration problem of pin-loaded holes. For the determination of solutions for the system of coupled partial differential equation (PDES) of the plate equation, (2) is equivalently converted into a single differential equation of the eighth-order in w_0. Expanding the method of complex-valued stress functions, which is well established in the theory of plane elasticity, the solution of the generalized plate equation (2) for multilayered composites is based on the complex displacement formulation for the homogeneous solution in case of purely mechanical stress

$$w_0 = 2\,\mathrm{Re}\left(\sum_{k=1}^{4} r_k \Psi_k\left(\mathfrak{z}_k\right) \right) \tag{4}$$

with four analytical functions $\Psi_k(\mathfrak{z}_k)$ referring to the four different complex planes $\mathfrak{z}_k = \mathfrak{z} + \lambda_k \bar{\mathfrak{z}}$ and $\mathfrak{z} = x + iy$.

The complex parameters λ_k are calculated as roots of the characteristic equation, which results from inserting (4) into the converted differential equation in w_0. Since in the case of real materials it can be shown, that from the eight different roots of the characteristic equation always two by pairs have to be conjugated complex, only four independent roots have to be taken into account.
For the effective handling of boundary conditions for the stress concentration problem, the notched area is projected onto the exterior of a unit circle using the method of conformal mapping. This opens the possibility of a uniform approach for the determination of the respective displacement functions, independently of the actual notch contour. In this special case the conformal mappings of the unit circle E in the ζ-plane onto the area of the plate S in the $\mathfrak{z}$-plane and the assigned affinely distorted areas $S^{(k)}$ in the respective $\mathfrak{z}_k$-planes are determined, where the points A, $A^{(1)}, \ldots, A^{(4)}$, which are assigned to each other by the affine projections, must have the same pre-image A_ζ on the edge of the unit circle E (see Figure 1) [6].

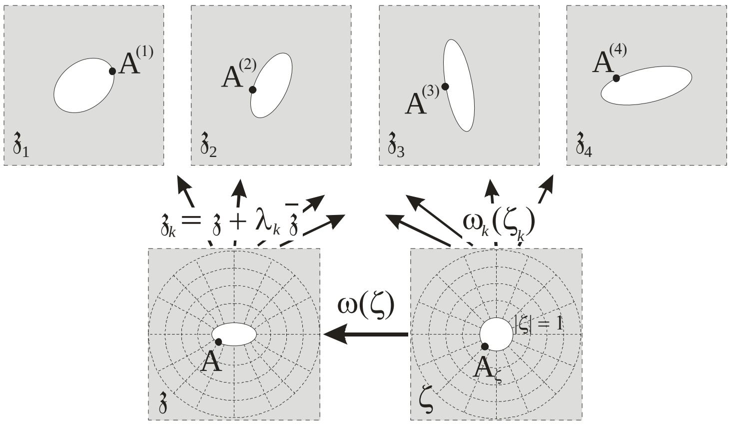

Figure 1: Conformal mappings of the ζ-plane onto the $\mathfrak{z}$- and $\mathfrak{z}_k$-planes

In the case of circular or elliptical cut-outs, covered by this publication, these mappings are given by

$$\mathfrak{z} = \omega(\zeta) = \frac{a+b}{2}\zeta + \frac{a-b}{2}\frac{1}{\zeta} \tag{5}$$

$$\mathfrak{z}_k = \omega_k\left(\zeta_k\right) = \left(\frac{1+\lambda_k}{2}a + \frac{1-\lambda_k}{2}b \right)\zeta_k + \left(\frac{1+\lambda_k}{2}a - \frac{1-\lambda_k}{2}b \right)\frac{1}{\zeta_k} \tag{6}$$

with $\zeta \equiv \zeta_k$ and a, b as the semi-axes of the elliptical notch.

2.3. Boundary conditions

In order to take into consideration the stresses resulting from the pin-loaded hole and additional stresses resulting from technically relevant external loads, the actual state of stress is decomposed using the superposition principle into three states of stress (Figure 2) as follows:

- **I**: a finite, unnotched plate with loads at the outer edge,
- **II**: an infinite notched plate with loads at the edge of the notch, adapted in such a way that, with superposition of I and II, an overall unloaded notch edge results,
- **III**: an infinite notched plate with the external loads at the notch edge.

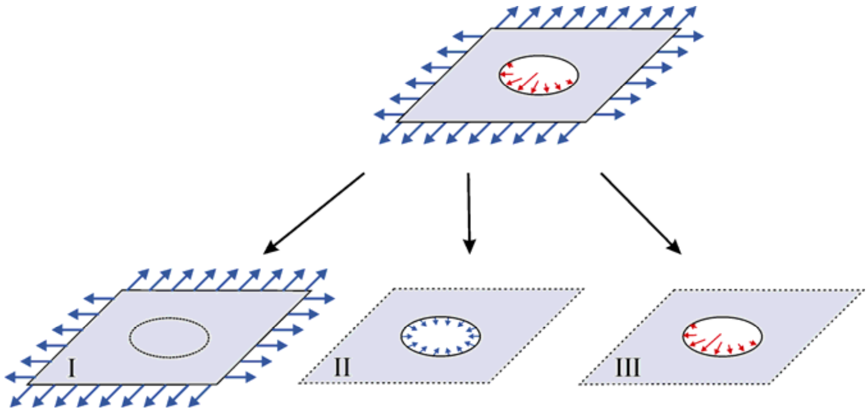

Figure 2: Decomposition of the coupled membrane-plate problem by means of superposition

The superposition of subproblems I and II corresponds to the problem of a notched anisotropic multilayered composite plate with unloaded notch edge. For this case the known solution methods are applied, for instance according to [5, 8].
In case of subproblem III, the pin-load is approximated by a sinusoidal distribution of normal edge forces on a continuous part of the notch edge (partial edge load), as frequently described in the literature (for instance [10]). The edge of the cut-out is divided into two disjoint subsections Γ_{OB} and Γ_{SB}. Assuming the natural parameterisation of the edge $\Gamma = \Gamma(s)$ with the edge parameter $s \in [0, 2\pi[$, the boundary condition for the edge can be written as

$$\tilde{N}_n(s) = \begin{cases} -p_0 \cdot \sin\left(\pi \cdot \dfrac{s-\vartheta_0^B}{\vartheta_1^B - \vartheta_0^B}\right) & \text{für} \quad s \in \left[\vartheta_0^B, \vartheta_1^B\right[= \Gamma_{SB} \\ 0 & \text{für} \quad s \in \Gamma_{OB} = \Gamma \setminus \Gamma_{SB} \end{cases} \tag{7}$$

with the force resultant amplitude p_0 (Figure 3).

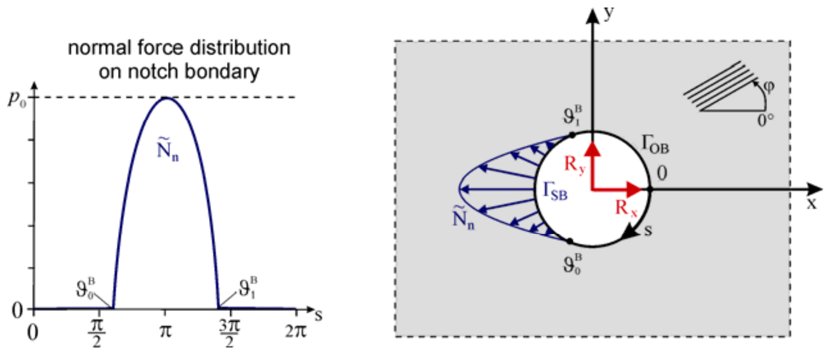

Figure 3: Cut-out with partial, sinusoidal normal forces at the notch edge

With a load specified in this way, it is obvious that the static equilibrium conditions on the overall plate are no longer fulfilled automatically. Therefore a virtual resultant $R = (R_x, R_y)^T$ is introduced as an auxiliary variable inside the cut-out, (see Figure 3, also refer to [3, 5])

$$R = \begin{bmatrix} R_x \\ R_y \end{bmatrix} = -\oint \begin{bmatrix} \tilde{N}_x \\ \tilde{N}_y \end{bmatrix} = \oint \begin{bmatrix} -\tilde{N}_n(s)\dfrac{dy}{ds} \\ \tilde{N}_n(s)\dfrac{dx}{ds} \end{bmatrix} \tag{8}$$

For the determination of the analytical displacement functions $\Psi_k(\mathfrak{z}_k)$ the formulations applied for subproblem II are supplemented by further terms $A_k \ln \zeta_k$, required due to the introduction of the virtual resultant R

$$\breve{\Psi}'_k(\mathfrak{z}_k) = \breve{\Psi}'_k(\omega_k(\zeta_k)) = A_k \ln \zeta_k + \sum_{m=0}^{\infty} B_{km}\zeta_k^{-m} \quad (k = 1 \ldots 4). \tag{9}$$

These logarithmic-terms $A_k \ln \zeta_k$ lead to a multi-leaf solution with ζ_k rotating around the notch. In order to determine the complex constants A_k, four additional boundary conditions (see [5]) are applied, which are fulfilled identically in the case of cut-outs without edge load, respectively with homogeneous edge load.

After this, a linear system of equations is created for determining the unknown coefficients A_k, B_{km} by complex Laurent series expansion, a comparison of coefficients on the notch edge and an additional boundary condition corresponding to the requirement of the existence of a reference point.

3. COMPARISON TO NUMERICAL RESULTS

For the verification of the developed analytical calculation method for multilayered composites with pin-loaded holes, extensive comparative finite element calculations are performed for CFRP multilayered composites and composites made of bidirectionally reinforced knitted glass fibre/polypropylene (GF/PP) fabrics with different lay-ups and different sectors of force transmission. An example for a FE model used in this investigation is given in Figure 4.

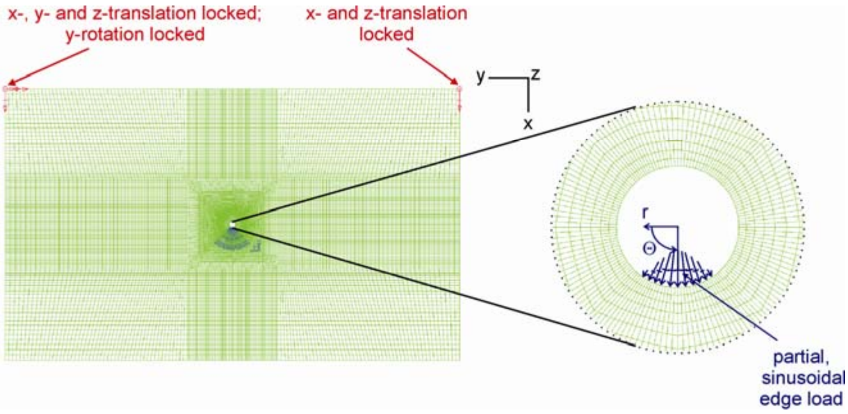

Figure 4: FE model with partial, sinusoidal edge load

Some representative results of the comparison of numerically determined and analytically calculated stress distributions on the edge of the notch are given in the following from the large number of performed verifying calculations. The composite plate is made of CFRP UD-layers (parameters: $E_{\parallel}$ = 135 GPa, $E_{\perp}$ = 8.2 GPa, $G_{\parallel\perp}$ = 4.7 GPa, $\nu_{\parallel\perp}$ = 0.3). The transmission of the partial, sinusoidal normal edge force with an amplitude of p_0 = 20 N/mm takes place through continuous parts of the notch-edge of $\Delta\vartheta$ = 180°, 90° or 45° (see Figure 3). Overall, it turns out that in the case of all examined combinations of composites and loads, the numerical results and the results obtained by the developed analytical methods show a high degree of agreement.

Figure 5 shows, for a symmetrical CFRP laminate, the normal, tangential and shear stresses for each layer with the edge forces being transmitted at $\Delta\vartheta$ = 90°. The normal and tangential stress curves of the -45°-layer can be obtained from the corresponding stress curves of the +45°-layer by reflection about a vertical axis through 180°, while the corresponding shear stresses can be obtained from a point-reflection about (180°, 0 MPa). This behaviour corresponds to expectations, due to the symmetrical structure of the laminate.

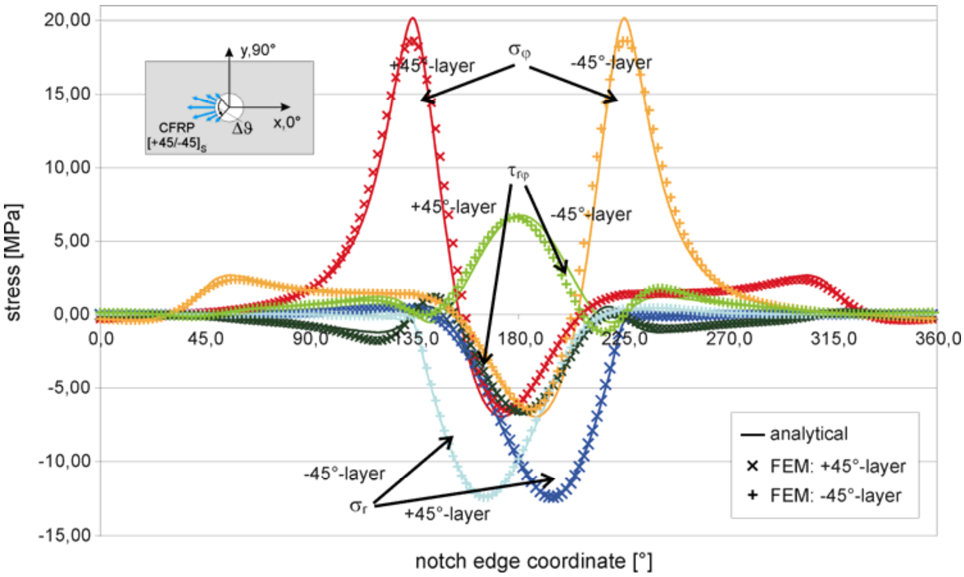

Figure 5: Analytical and numerical stress curves of a notched MLC plate (CFRP, $[+45/-45]_s$) with partial edge load ($\Delta\vartheta$ = 90°).

The maximum of the tangential stresses at the notch edge can be found in the vicinity of the points where the fibre orientation of the individual layer is tangent to the edge of the cut-out, i.e. for the +45°-layer at 135° and, correspondingly, for the -45°-layer at 225°. The layer-by-layer comparison of stress curves also shows the distribution of load bearing performance to the individual layers. It becomes apparent that the layer-related stress and distortion analysis is of fundamental importance for a detailed evaluation of material strain.

Figure 6 shows the corresponding stress curves for an unsymmetrical [0/90] CFRP multilayered composite with an edge load transmission at $\Delta\vartheta$ = 90°. The stress values are determined for the geometrical midplane of the individual lamina.

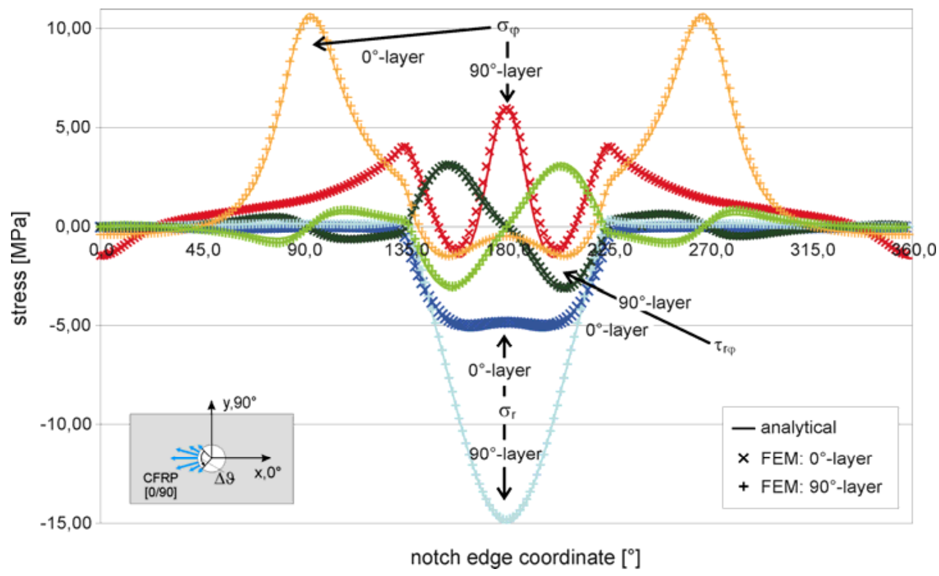

Figure 6: Analytical and numerical stress curves of a notched MLC plate (CFRP, [0/90]) with partial edge load ($\Delta\vartheta = 90°$).

For this unsymmetrical multilayered composite the numerical and analytical results also show a high level of agreement. In this cross-ply composite it is once again noticeable that for the 0°-layer the extreme values of the tangential stresses are reached in the areas where the fibre orientation is tangent to the notch edge, i.e. at 90° and 270°. For the 90°-layer however, the tangential stress curves show a different behaviour.

As an explanation, Figure 7 shows a comparison of tangential stress curves for the 90°-layer at different force transmission sectors $\Delta\vartheta$. While for a transmission of normal edge forces at $\Delta\vartheta = 180°$ the tangential stress, as expected, rises in the area around 180° to an absolute maximum, this stress peak diminishes with a decreasing force transmission sector and between $\Delta\vartheta = 90°$ and $\Delta\vartheta = 45°$ even enters the compressive range. These compressive stresses in case of small force transmission sectors are above all caused by transversal contraction and by the impediment to deformation by the 0°-layer.

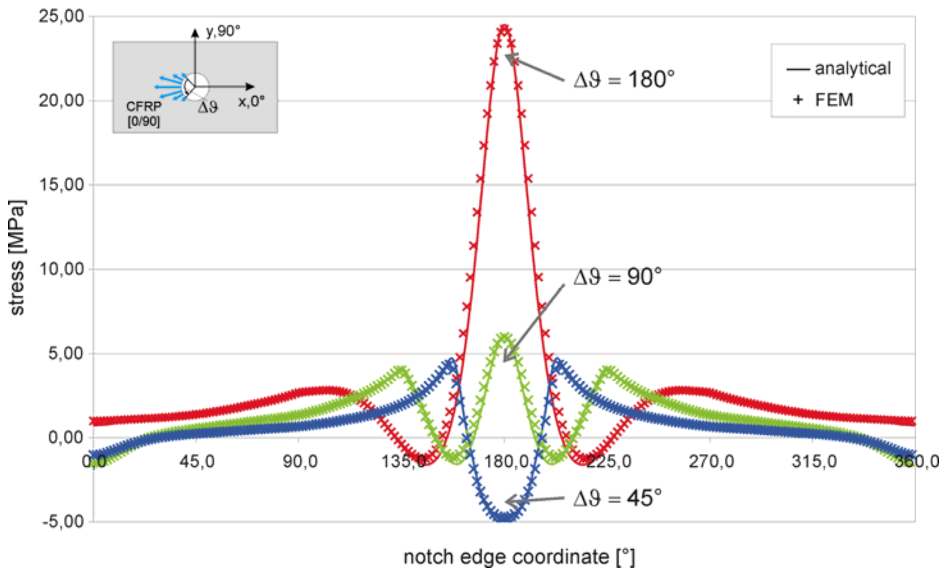

Figure 7: Analytically and numerically determined tangential stress curves for the 90°-layer of a notched MLC plate (CFRP, [0/90]) with partial edge load at $\Delta\vartheta$ = 180°, 90°, 45°.

In summary, a high level of agreement is determined between analytically calculated and numerically determined stress curves for all examined parameter combinations. Thus, the developed fundamentals for the stress concentration analysis of notches with a partial, sinusoidal normal edge load represent a quick, dependable and easily manageable alternative to performing time-consuming FE analyses when dealing with pin-loaded holes. In particular, they allow to identify unusual phenomena – for instance, local change of sign of tangential stresses when changing the sector of force transmission – in an early design phase and to take advantage of them deliberately in the optimisation of composites.

4. PARAMETER STUDIES ON MULTILAYERED COMPOSITE PLATES WITH PIN-LOADED HOLES

A parameter study on the influence of the pin diameter is shown here as an example of a sensitivity analysis. Knowledge of this influence on the notch stress field in MLC plates is of great importance for the evaluation of, for instance, pin, screw or rivet connections within the scope of an application and component design in line with material characteristics.

4.1. Model for parameter variation

The continuous part of the notch-edge $\Delta\vartheta = \vartheta_2 - \vartheta_1$, through which the loaded pin with a radius of r_i (i = 1,2. . .) transmits forces into the MLC plate, is regarded as a variational parameter (Figure 8). Here, it is assumed that this sector of load transmission lies symmetrically to the *x*-axis. For the purpose of this examination, the load *F* acting upon the pin is assumed to be constant. The sinusoidal partial normal edge load as per (7) is used as an approximation for the load transmission into the MLC plate by the loaded pin.

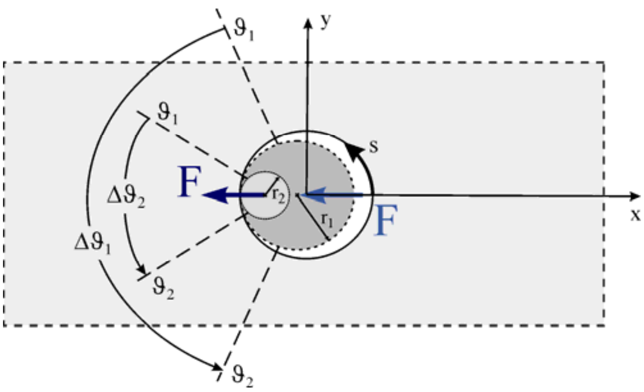

Figure 8: Model for the examination of the influence of the pin diameter

4.2. Results of parameter studies

The normal, tangential and shear stress distributions at the edge, depending on $\Delta\vartheta$, are calculated for the individual layers and graphically displayed in form of 3D-diagrams with the aid of a software tool that was developed within the scope of the examinations. The determination of stresses takes place with reference to the geometrical midplane of the individual laminae. In Figure 9 the results of the parameter study for a symmetrical $[+45/-45]_s$ CFRP composite with a total thickness of 2 mm are given.

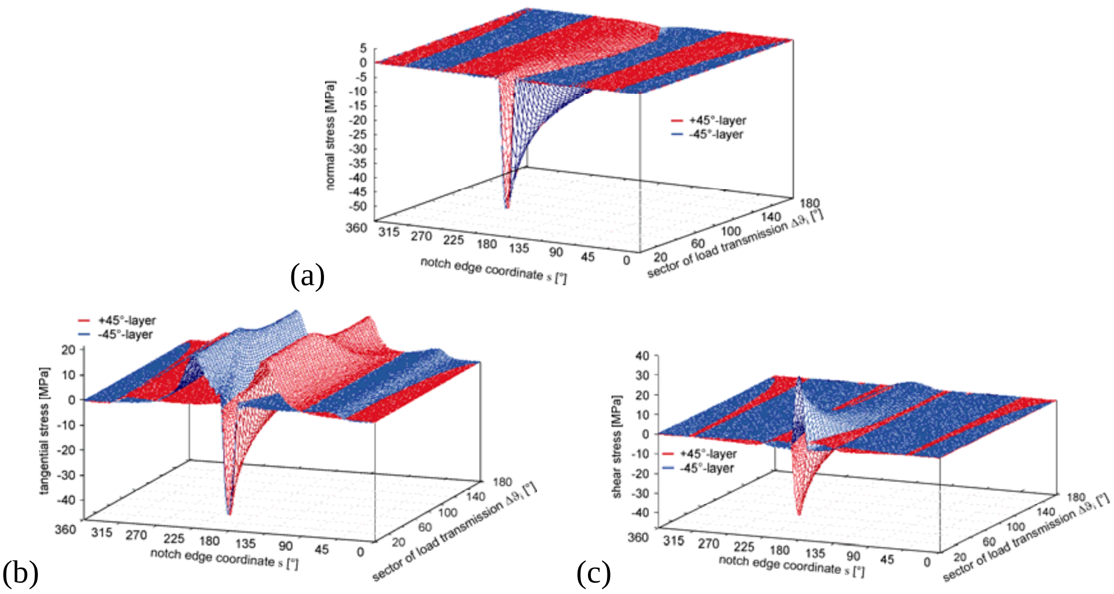

Figure 9: Distribution of edge stresses for each layer depending on the load sector; MLC plate: CFRP $[+45/-45]_S$: (a) normal stress, (b) tangential stress and (c) shear stress

The diagrams clearly show how the edge stresses are distributed to the individual layers and which layer assumes the main load-bearing function, depending on the notch edge coordinates. Particularly in case of the tangential stresses (Figure 9(b)) this can be seen very clearly. The symmetry of the composite lay-up is also reflected in the diagrams.
It is apparent that all three stress components may reach failure-critical magnitudes, which again emphasizes that a comprehensive layer-by-layer determination and evaluation of dimensioning values is indispensable for a component design in line with material characteristics.

5. CONCLUSIONS

The problem of pin-loaded holes occurs as a subproblem of many technically relevant problems especially dealing with bolt joints, rivets, screws or other fasteners. The mechanically equivalent model of a cut-out, which is subject to a partial sinusoidal normal edge load, has been dealt with here by means of analytical methods for the first time for a layer-by-layer analysis of multilayered composites. It should be emphasized that the presented methods not only calculate stresses, strains and displacements directly at the edge of the notch, but also their distribution throughout the entire area of the plate. A vast number of FE analyses were performed with symmetrical and unsymmetrical composite structures for the verification of the developed calculation methods. A comparison of numerical results and the results obtained by means of the developed analytical solutions show a very good agreement. The performed parameter studies demonstrate that very complex mechanisms are acting upon pin connections of anisotropic MLC plates in the area of notches. Therefore a precise layer-by-layer analysis of the distribution of stresses and distortions is absolutely indispensable in the interest of dimensioning such connections in line with material and component characteristics. Starting from this layer-by-layer approach the development of new failure criteria for notched orthotropic composites using so called physically based failure criteria and under additional consideration of the so-called microsupport effect according to NEUBER [11] is currently one of the focal points of ongoing research work at the ILK.

6. ACKNOWLEDGEMENTS

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7. REFERENCES

1- Hufenbach W., Grüber B., Langkamp A., Lepper M., Kroll L., “Advanced calculation methods for notched hybrid composites with textile-reinforced polymers”, *Journal of Plastics Technology*, 2007;3.

2- Engels H., Becker W., “Closed-form analysis of external patch repairs on laminates”, *Composite Structures*, 2002;56:259-268.

3- Lekhnitskii S. G., “*Anisotropic Plates*”, Transl. 2nd russ. edition: S. W. Tsai and T. Cheroin, New York et al.: Gordon and Breach, 1968.

4- Echavarría C., Haller P. and Salenikovich A.,“Analytical study of a pin-loaded hole in elastic orthotropic plates”, *Composite Structures*, 2007;79:107-112.

5- Grüber B., “*Beitrag zur Strukturanalyse von anisotropen Schichtverbunden mit elastischen Einschlüssen und Bolzen*”, PhD-Thesis, Technische Universität Dresden, 2004.

6- Grüber B., Hufenbach W., Kroll L., Lepper M., Zhou B., “Stress concen-tration analysis of fibre-reinforced multilayered composites with pin-loaded holes”, *Composites Science and Technology*, 2007;67:1439-1450.

7- Zhou B., „*Beitrag zur Berechnung endlich berandeter anisotroper Scheiben mit elastischem Einschluss*“, PhD-Thesis, Technische Universität Clausthal, 1997.

8- Lepper M., „Kerbspannungsanalyse anisotroper Mehrschichtverbunde mit symmetrischem und unsymmetrischem Strukturaufbau“, PhD-Thesis, Technische Universität Clausthal, 1999.

9- Hufenbach W, Kroll L., “Stress analysis of notched anisotropic finite plates under mechanical and hygrothermal loads”, *Archive of Applied Mechanics*, 1999;69:145-159.

10- Chang F. K., Scott R. A., Springer G. S., “Strength of mechanically fastened composite joints”, *Journal of Composite Materials*, 1982;16:470-494.

11- Neuber H. „*Kerbspannungslehre*“, Berlin: Springer-Verlag; 2001.