PLANE STRESS WAVE INTERATION UNDER IMPACT ON THE SURFACE OF A LAYERED COMPOSITE PANEL

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ABSTRACT

The interaction of plane tensile/compression waves transmitting within a sandwich panel normally to its surface is considered. Analytic modeling presents a modification of the method of characteristics for one – dimensional wave equation in the problems of the impact over plate surface with a rigid body. Displacements, velocities and stresses are determined through the initial boundary velocity and the stationary force field of the striker mass and the plate mass. The method of analytic continuation in time put forward carries out stress analysis for an arbitrary time interval from the finite expressions. Contrary to the frequency stress analysis which is in common used with harmonic decomposition of disturbances in series, the approach advanced carries out stress analysis considering the first displacement derivative discontinuity and escaping calculation of discontinuities in summation of series.

The transmission and reflection coefficients of the pressure pulses on the contact surfaces of layers with different physical properties are determined. The expressions for tensile stresses in the panel face layers and filler, which are responsible for the material failure by spalling, are presented. The stresses in relation to the geometry and dynamic parameters of the sandwich structure are analyzed. In the case of a symmetric panel structure, the stress pattern in the midlayer and on its contact boundaries is given, which takes into account the branching and superposition of pulses.

1.INTRODUCTION

Numerous model constructions for studying dynamic effects in the heterogeneous structures confirm that it is difficult to secure an adequate model description of dynamic phenomena in composites. The effect of stress waves on the spalling fracture of the surface layer with reference to a sandwich panel has been calculated usually through numerical methods [1]. Dynamic fracture relating to the surface delaminations of laminate plates has been specified as a kind of bifurcation under quasistatic loading in [2].

In this study the approach to the study of the wave process in laminate structure is developed for determining the stresses, displacements, and velocities of points under impact pulsed loading. The method of analytic continuation in time put forward carries out stress analysis for an arbitrary time interval from the finite expressions. The first cycle of wave propagation is chosen because the displacements and stresses arising at the instant a short-time pulse passes through a point of medium are analytically simple and can be determined exactly. In this special case the initial velocity is determined on account of short time action till loading mass bounces from the plate. Taking into account the complex pattern of pulse branching across the thickness of the laminate structure, the analysis of transverse stresses is restricted here to a three-layered structure of sandwich-type. To obtain analytical dependencies for the quantities to be calculated, the one-dimensional problem was generalized to the case of conjugated media with different properties. This aggregate method may be inconvenient in studying steady-state vibrations, since not always does it allows one to detect the natural frequencies in a simple way. However, the method of separation of variables (the Fourier method), which is most convenient for determining the natural frequencies of steady-state vibrations, is too labor-consuming for analyzing the stresses, because their expansion in modes each of which represents a standing wave requires a large number of terms of the series for the time-discontinuous passage of pulses through a point of the medium.

2.WAVE MOTION IN A FINITE BAR UNDER IMPACT ACTION MASS AND GRAVITATIONAL FIELD

In general the basic stress state of a plate under plane wave motion in the line of a normal consist of “side” normal stresses acting in the plane of a plate in addition to transverse stress $\sigma_z(z,t)$ as it should be allowed for [3]. With no displacements over the plane the “side” stresses are dependent on material Poisson's ratio. Their inclusion in calculations at rigid fixing the end faces of the plate makes the decision procedure essentially complicated. Simplification of the stress state calculation is rendered possible at the cost of reduction to one - dimensional model with zero value of Poisson's ratio, or just assuming the plate deformability solely across its thickness, as is the case for deformability along the length in the theory of bars. In doing so the problem set up for a plate under plane wave action is reduced to a bar model. Symbol h for the plate thickness is replaced with the symbol l for the bar length.

Now the method of characteristics can be presented for obtaining the closed solution of wave equation with the differential designation of the condition at the end of a bar $x = l$ including the impact on its end with the action of gravitational force field in the wave process. The known previously solution of this classical problem is based on the series of trigonometric functions including the coefficients dependent on fundamental frequency infinite spectrum [1], [3]. The initial iteration cycles by the method of characteristics have been undertaken in [4], [5].

The statement of the wave problem for a bar fixed at one end $(x = 0)$ with the other free end $(x = l)$ subjected to the impact of the axially directional load M at initial time $t = 0$ is reduced to the solution of inhomogeneous equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = g \tag{1}$$

and boundary conditions to the displacement of one bar end as well as balance of the forces applied to the other end

$$u\big|_{x=0} = 0\,, \qquad \left.\frac{\partial^2 u}{\partial t^2}\right|_{x=l} = g - \frac{a^2}{ml}\left.\frac{\partial u}{\partial x}\right|_{x=l} \qquad \text{at} \quad t \ge 0 \tag{2}$$

Initial displacement and velocity distribution are written as

$$u(x,0) = \frac{gx(2l - x)}{2a^2} \qquad \text{at} \qquad 0 \le x \le l$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \quad \text{at} \quad 0 \le x \le l\,, \qquad \frac{\partial u}{\partial t} = -v \quad \text{at} \quad t = 0 \ \text{and} \ x = l \tag{3}$$

where g - acceleration of gravity, $m = \frac{M}{\rho S l}$, ρ, S - specific density and cross-section area of a bar l in length. The speed of sound is designated by $a = \sqrt{E/\rho}$.

The condition of fixed boundary at $x = 0$ reduces two unknown functions for the displacement $u(x,t)$ to the only function and the solution becomes

$$u = \varphi(at - x) - \varphi(at + x) + \frac{gx(2l - x)}{2a^2} \tag{4}$$

The initial conditions (3) at $t = 0$ demonstrate the function $\varphi(x) = \varphi(-x)$ and its derivative $\varphi'(x) = \varphi'(-x)$ to be the even functions at least on open interval $0 < x < l$. Then over the interval $\varphi(x) = Const$ is the case only and with the first condition in (2) $Const = 0$ can be taken. Note that the argument of the even function considered here ought to be any number falling into the interval $(-l, l)$. The argument is further assumed as the number z taking the value of binomial $at - x$ or $at + x$. Unlike complex number z here both items of the binomials are real numbers, and subtracting x corresponds to propagation of a forward wave along the x axis while the added x - to back wave oppositely directed. So the initial function is

$$\varphi(z) = 0 \quad \text{at} \quad -l < z < l \tag{5}$$

With the aim of analytical continuation of the function $\varphi(z)$ beyond the domain of the argument indicated in (5), that is definition $\varphi(z)$ at $z \geq l$, the second boundary condition (2) is used which allowing for (4) can be rewritten in terms of arguments $at + l$ and $at - l$. Setting the first argument varies depending on time t as $z = at + l$ the key equation can be derived from (2)

$$\varphi''(z) + \frac{1}{ml}\varphi'(z) = \varphi''(z - 2l) - \frac{1}{ml}\varphi'(z - 2l) - \frac{g}{a^2} \tag{6}$$

in which all primes will be applied to differentiation with respect to z . The equation enables the function $\varphi(z)$ to be continued beyond the boundaries of the interval $(-l, l)$. Note here again that the function φ defined from (6) is independent of the “origin” of the argument z and its domain boundaries are what matters solely. The limitless growth of the argument under limited bar length l is actually occurred through extension of time. In this respect the argument z is a “time” that. Moreover any combination of the time and spatial coordinates represented as $z = at \pm x$ and falling into the domain of a given function φ offers the argument thereof.

The integral of the equation (6) at $l \leq z \leq 3l$ should present no problems and when taken into account that $\varphi(z - 2l) \equiv 0$ as well as $\varphi(l) = 0$, $\varphi'(l) = v/a$, we obtain:

$$\varphi_1(z) = -\frac{gml}{a^2}(z - l) + ml\left(\frac{v}{a} + \frac{gml}{a^2}\right)\left(1 - e^{-\frac{z-l}{ml}}\right)$$

$$\varphi_1'(z) = \left(\frac{v}{a} + \frac{gml}{a^2}\right)e^{-\frac{z-l}{ml}} - \frac{gml}{a^2}, \quad l \leq z < 3l \tag{7}$$

In the context of our further interest in a quick recoil due to impact pulse, we shall deduce limitations on the initial velocity and mass of a striker such that its separation from the bar takes place in the time interval less than one cycle duration. In this case the impact pulse arising from stresses at the contact surface is only dependent on characteristic $at+l$ referring to primary incident wave as the back wave occurring in the first cycle only after reflection from the fixing point will not reach yet the impact mass. With (13) can be taken:

$$J=\int_0^{t_1^*}\sigma\big|_{x=l}dt=-\int_0^{t_1^*}\varphi_1'(at+l)dt=\frac{gml}{a^2}\Big[1+\Big(1+\frac{av_0}{gml}\Big)\Big(1-e^{-\frac{at_1^*}{ml}}\Big)\Big] \tag{8}$$

At given striker and bar mass ratio and time of their contact the limitations on the initial velocity shall be derived from the absence of compression at a contact point for the instant t_1^*, that is $u_x'(x=l)=0$. From the latest condition on account of (4), (7) follows $t_1^*=(ml/a)\ln(1+\chi)$. If we propose to put the time of quick recoil $t_1^*<2l/a$, the limitation on mass ratio at a given velocity is imposed by the following constraint

$$m\ln(1+\chi)<2\,,\qquad \chi=\frac{av_0}{mgl} \tag{9}$$

Exponentiation (9) gives the limitation on velocity at a given mass ratio

$$v_0<\frac{mgl}{a}(e^{\frac{2}{m}}-1) \tag{10}$$

Should the interval $2l/a$ is multiple contact time, than $t_1^*=\dfrac{2l}{ak}$ and at $k=1,2,..$ the duration of an impact pulse causes the reduction in the initial velocity

$$v_0=\frac{mgl}{a}\big(e^{\frac{2}{km}}-1\big),\qquad k=1,2,... \tag{11}$$

With a mutual reduction in m proportionally with the fall in contact time the initial velocity of impact will be reduced in equal ratio.

The initial stress at the contact surface with area S by (11) equals

$$\sigma_0=-v_0\sqrt{E\rho}=-\frac{Mg}{S}\big(e^{\frac{2}{km}}-1\big),\qquad k=1,2,... \tag{12}$$

The velocity v_o in (12) is taken with minus sign as the bar is impacted upwards on its lower end. Should we change the direction of impact and displace a fixing point to a lower end the inequality (9) at $\chi<0$ does not take place and the mass does not bounce off on the first time cycle.

It is worthy of note here that the solution of the similar impact problem with open back surface (first condition (2) is substituted by a free end condition

$u_x(0,t)=0$) leads to identical results for the quick recoil time of mass over the first cycle. As a consequence, any degree of fixity of a back surface – from zero to infinity - may be thought of as leaving the parameters of a quick recoil on the first cycle unchanged. In the event that a plane impact occurs normal with respect to a layer system, the pulse may be approximately substituted for well-defined stress σ_0 acted for a time $t_1^* = \frac{2l}{ak}$ and calculated from (12) at reasonably large k. The higher approximation to the stress as generating the pulse (8) is obtainable by averaging (8) over the time right up to the recoil and taking into account the velocity (11)

$$<\sigma_0> = J / t_1^* = -\sigma_{st} [\frac{km}{2}(e^{\frac{2}{km}} - 1) - 1] \tag{13}$$

where static stress $\sigma_{st} = Mg / S$. Expanding (12) and (13) into a power series at larger k it is easy to verify that $\sigma_0 / <\sigma_0> = 2$ and $<\sigma_0> = -gl\rho / k$. The stress can be taken into account in strength analysis under vibrations when m ratio is sufficiently small and accordingly the velocity of a striker is sufficiently large. If not, at the impact up to the recoil of a striker, the stresses in the cycles which follow the first cycle should be calculated. From here on for purposes of investigating the ramification of a pulse applied over entire surface of layer structure, the initial velocity is considered to be reasonably large and in accordance with the order of magnitude $\sigma_* / \sqrt{E\rho}$ where σ_* is the ultimate stress of material.

3. TRANSLATION WAVE IN TWO SEMI-INFINITE HYBRID BARS

For a wave process in two semi-infinite bars with different properties, contacting on the section $x = 0$, the equations are identical with (1) at the mention of index $i = 1,2$ for displacements $u_i(x,t)$ and coefficients a_i; else taken $g = 0$. Here, we assume that $-\infty < x < 0$ at $i = 1$ and $0 < x < \infty$ at $\text{i} = 2$. Let the displacements of points of the left-hand region are defined by a function of difference between the time and the spatial coordinates, $f(t - x / a_1)$, at $t \le 0$. The initial functions of the wave processes (at $t = 0$) in the two dissimilar regions are written as

$$u_1(x,0) = f(-x / a_1), \quad \frac{\partial u_1(x,0)}{\partial t} = f'(-x / a_1), \quad -\infty < x < 0,$$

$$u_2(x,0) = 0, \quad \frac{\partial u_2(x,0)}{\partial t} = 0, \quad 0 < x < \infty. \tag{14}$$

At $t = 0$, a backward wave reflected from the boundary $x = 0$ of the right-hand region starts to move into the left-hand region $x \le 0$; in the right-hand region, only the forward wave propagates, as having passed through the contact plane into a region with a zero initial disturbance. Therefore we can write

$$u_1(x,t) = \theta_1(t - x / a_1) + \theta_2(t + x / a_1), \qquad u_2(x,t) = \vartheta(t - x / a_2) \tag{15}$$

Taking the continuity of displacements and stresses as contact condition on the boundary $x = 0$ between the two bars,

$$u_1(0,t) = u_2(0,t), \qquad E_1 \frac{\partial u_1(0,t)}{\partial x} = E_2 \frac{\partial u_2(0,t)}{\partial x}, \quad t \geq 0, \tag{16}$$

after substituting in them the functions from (15), we obtain the following equations:

$$\theta_1(t) + \theta_2(t) = \vartheta(t), \qquad \sqrt{E_1 \rho_1}[\theta_1'(t) - \theta_2'(t)] = \sqrt{E_2 \rho_2}\, \vartheta'(t) \tag{17}$$

We should note that equations (17) are written for revealing an interrelation between the functions θ_1, θ_2 and ϑ in time, but these equations remain the same for any other argument linearly connected with the time $\xi = t \pm x / a_i$. Thus, we have from (17)

$$\theta_2(\xi) = \frac{\sqrt{E_1 \rho_1} - \sqrt{E_2 \rho_2}}{\sqrt{E_1 \rho_1} + \sqrt{E_2 \rho_2}} \theta_1(\xi) - C, \qquad \vartheta(\xi) = \frac{2\sqrt{E_1 \rho_1}}{\sqrt{E_1 \rho_1} + \sqrt{E_2 \rho_2}} \theta_1(\xi) + C \tag{18}$$

The constant C must be set equal to zero at zero displacement of the two bars as solid bodies. The final form of the functions for the forward waves of displacements, $\theta_1(t - x / a_1)$ and $\vartheta(t - x / a_2)$, and for the backward wave, $\theta_2(t + x / a_1)$, is determined from initial conditions. After substituting the function $\theta_1(t - x / a_1)$ at $t = 0$ in (14) and taking into account the definition of the function ϑ by (18), we have

$$\theta_1(-x / a_1) = f(-x / a_1), \quad x < 0, \text{ and } \theta_1(-x / a_2) = \theta_1'(-x / a_2) = 0, \quad x > 0 \tag{19}$$

As follows from (19), $f(x) = 0$ at $x < 0$. As a result, we obtain the displacement functions in the two regions examined

$$u_1(x,t) = f\big(t - \frac{x}{a_1}\big) + \frac{\sqrt{E_1 \rho_1} - \sqrt{E_2 \rho_2}}{\sqrt{E_1 \rho_1} + \sqrt{E_2 \rho_2}} f\big(t + \frac{x}{a_1}\big), \quad -\infty < x < 0, \quad t > 0,$$

$$u_2(x,t) = \frac{2\sqrt{E_1 \rho_1}}{\sqrt{E_1 \rho_1} + \sqrt{E_2 \rho_2}} f\big(t - \frac{x}{a_1}\big), \quad 0 < x < \infty, \quad t > 0 \tag{20}$$

4.TRANSVERSE IMPACT ON THE SURFACE OF A SANDWICH PANEL

Let us now consider the wave process in a sandwich panel under a plane impact on its upper surface, when the lower surface is free (Fig. 1). At the initial moment $t = 0$ over a boundary $z = 0$ we shall give a specification to the stress as

$$\sigma_0 = E_1 \frac{\partial u_1}{\partial x}\bigg|_{t=0,\ x=-0} = -(E_1 / a_1) f'(0) = -\sqrt{E_1\ \rho_1}\ f'(0) \tag{21}$$

A pulse of compression acts on the boundary necessarily at the positive value of a derivative $f'(0)$, then the velocity $v_0 = f'(0)$ is positive. Let the velocity satisfies an equation (11). It is quite obvious, when taking into consideration (20) relating to the intermediate boundary at the moment $t = h_1 / a_1$, the stress and the velocity of a reflected wave (or reflected pulse designated further by a superscript R) and the pulse passing into the second layer (designated by a superscript T) will be related to σ_0 and v_0 by the following factors:

$$\sigma_1^{\ R} = \sigma_0 (K_2 - K_1)/(K_1 + K_2), \quad \sigma_2^{\ T} = 2\sigma_0 K_2 /(K_1 + K_2) \tag{22}$$

$$v_1^{\ R} = v_0 (K_1 - K_2)/(K_1 + K_2), \quad v_2^{\ T} = 2 v_0 K_1 /(K_1 + K_2) \tag{23}$$

where $K_i = \rho_i a_i = \sqrt{E_i \rho_i}$ is the dynamic stiffness of the layer of specific material.

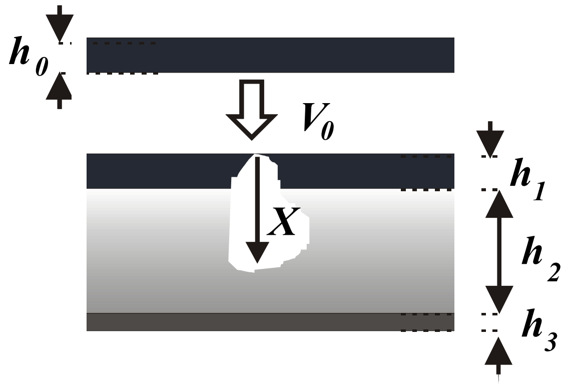

Figure 1. Typical structure of a sandwich panel exposed to a normal surface impact.

The following inference can be stated by dint of analysis of the factor combination in the expressions for stress and velocity (22), (23). The transmission of a pulse of pressure through the boundary of two layers will be either partially blocked at passage into less rigid layer with the increment in velocity of particles v (in this case $\operatorname{sgn} \sigma_1^R = -\operatorname{sgn} \sigma_1^0$, and $\left|\sigma_2^T\right| < \left|\sigma_1^0\right|$, $v_2^T > v_1^T$), or amplified as a velocity is reduced at passage into more rigid layer (in this case $\operatorname{sgn} \sigma_1^R = \operatorname{sgn} \sigma_1^0$ and $\left|\sigma_2^T\right| > \left|\sigma_1^0\right|$, $v_2^T < v_1^T$) depending on the ratio of the dynamic rigidities K_1 and K_2. The reflected pulse is absent when the value K_i is independent of i and in this case the transmitting pulse is equal to the initial one, marked by a zero. In case of reflecting from an absolutely rigid layer, that is $K_2 = \infty$, we receive corresponding absolutely plain result $\sigma_1^{\ R} = \sigma_0$, $\sigma_2^{\ T} = 2\sigma_0$, $v_1^{\ R} = -v_0$, $v_2^{\ T} = 0$, that follows from (22), (23); and when the contact surface is a free surface that is in the absence of the second medium $(K_2 = 0)$ we obtain $\sigma_1^{\ R} = -\sigma_0$, $\sigma_2^{\ T} = 0,$ $v_1^{\ R} = v_0$, $v_2^{\ T} = 2v_0$. It is precisely the last case that will characterize the wave process spreading up to a back surface - the lower face plane of a sandwich plate. The pressure pulse being blocked and branched in layers changes its sign in the last lower layer. The estimation of the tensile stresses inevitably appearing due to reflecting from a free surface and converting in all layers is of interest in an effort to predict a slabbing failure.

When the falling pulse reaches the third layer, the tensile stress reflected from the back surface was marked as σ_{1-3}^R. Then the stress from the tension pulse, which aroused in the third layer and returned to the first layer, can be written as σ^{T}_{1-3-1}. Having used (22) these stresses can be written:

$$\sigma^{R}_{1-3} = -4\sigma_0\chi_3/(1+\chi_1)(1+\chi_3), \quad \sigma^{T}_{1-3-1} = -16\sigma_0\chi_1\chi_3/(1+\chi_1)^2(1+\chi_3)^2 \quad (24)$$

where $\chi_1 = K_1/K_2$, $\chi_3 = K_3/K_2$.Tensile stress σ^{T}_{1-3-1} can be greater than that in the lower layer, if, as follows from (24), the relative dynamic stiffness of the first layer χ_1 is high enough. Figure 2 shows these stresses as functions of the logarithmic values of the ratio between the dynamic stiffnesses in intervals symmetric relative to the zero value of the logarithm for a homogeneous plate. The asymmetry condition $\chi_1 >> \chi_3$ of the panel structure necessary for the stress σ^{T}_{1-3-1} to exceed the tensile stress in the back (third) layer is already accessible at $\log\chi_1 > 0$ and $\log\chi_3 < -1.5$, as follows from a comparison of the stresses in Fig. 2.

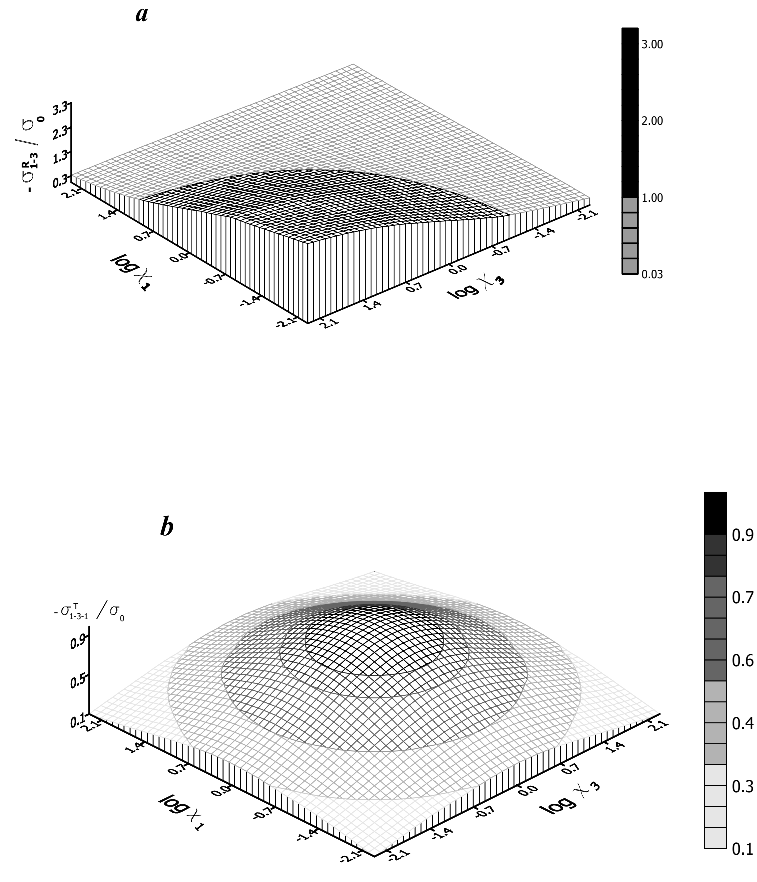

Figure 2. Relative stress in the third layer after reflection of a pressure pulse from the back surface of the panel (a) and its return into the first layer, (b), vs. the relative dynamic stiffness of the panel, $\chi_i = \sqrt{E_i\rho_i/E_2\rho_2}$, $i = 1,3$.

One could see from Fig. 2 b, remarkable for the constant level of stresses, the stacking sequence of a sandwich panel is designed through module of the sum of logarithmic relative stiffness squared: $\log^2\chi_1 + \log^2\chi_3 = R^2$. When comparing the positive values of stresses in the surface layers, we must also separate out the stress

σ_1^R formed upon reflection of the falling pulse from the contact surface between the first and second (not so stiff) layer if $\chi_1 > 1$. Since, in this case, the reflection coefficient $(1-\chi_1)/(1+\chi_1) < 1$, the reflected stress is tensile. Thus, the positive stresses $\sigma_1^R, \sigma_{1-3}^R$, and σ_{1-3-1}^T, arising as a result of transformation of the falling compression pulse across the thickness of the layered structure of the panel, can cause the spalling fracture of surface layers of the sandwich panel.

5. CONCLUSIONS

1. The method of characteristics giving an analytic solution continuation in time advanced here is applicable to the study of shock waves in bars and plates. The expressions arising from the closed form solution along the length l at zero time $t = 0$ have been mathematically constructed.
2. Variation factors are deduced for the longitudinal waves of displacements, velocities and stresses propagating in hybrid bars and transiting the contact section of two adjacent bars of different dynamic stiffness. These factors are applied to the analytical construction of relationships for branching and superposition of pressure pulses in the layers of a sandwich panel at impact loads.
3. The greatest stress depends on the relative dynamic stiffness of sandwich layers and their relative thickness. It was shown with respect to an asymmetric sandwich, the tensile stress in the rigid first layer that in the beginning has been exposed to an impact load, on a pulse return can exceed the stress in the third layer in which a tensile pulse was generated by the reflection of a pressure pulse from a back surface. This is not the case for a sandwich of symmetric structure.

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