Validation of a piezoelectric laminated plate modelling

J.F. Caron, M. Vandamme, A. Ehrlacher

NAVIER Institute / LAMI
Ecole Nationale des Ponts et Chaussées (ENPC),
Champs-Sur-Marne, Marne-La-Vallee, France
caron@lami.enpc.fr

ABSTRACT

A new model is presented for the problem of linear piezoelectric laminate under an applied surface traction or potential. According to the usual denomination employed in published works, the new model proposed here can be considered as a layerwise theory. The main aim of this simplified model is to predict approximately but easily the shear stress and the normal electric displacement at the interfaces of different layers. We are especially interested in predicting edges effects often responsible of damage in the laminate.
The comparisons with analytical solutions are very satisfying.

Index terms- piezoelectric - mixed-model - plates - edge effects.

1. INTRODUCTION

A high number of theories have been developed in the recent years for modeling piezoelectric multilayered beams or plates [5][8][9]. According to the usual denomination employed in published works, the new model proposed here can be considered as a layerwise theory, called Multiparticular Model of Multilayered Materials (M4).
This theory was developed previously to determine interlaminar stresses in multilayered non-piezoelectric composite plates [1],[2] often responsible of delaminations [3]. The originality of this theory is based on a stresses approach, and the use of the Generalized Hellinger-Reissner's mixed variational function. Some approximations are made about the z variations of the 3D stresses, but no kinetic approximation is made. The aim of the present work is to extend this formulation to piezoelectric multilayered materials. 3 mechanic generalized stress fields and 2 electric generalized fields of induction are introduced in each layer. 2 mechanic generalized stress fields and 1 electric generalized field of induction are introduced at each interface. The theory leads to the determination of generalized kinetic displacements and electric potential in each layer, and to the determination of a generalized deflection. The Hellinger-Reissner's variational principle gives generalized bounding conditions, equilibrium equations and constitutive relations. These equations take explicitly into account the coupling between electric and mechanic quantities. Moreover, they can serve to resolve analytically some piezoelectric problems.

2. GENERALIZED HELLINGER-REISSNER FORMULATION

The model is based on a generalized Hellinger-Reissner formulation of a 3D heterogeneous piezoelectric problem.
In such a formulation the coupled mechanical-electrical energy density is written in terms of the 3D stress field components σ_{ij} and electric displacement field components D_i. $\left(i, j \in \{1,3\}\right)$
The dual fields are the strain field components

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and the electrical field components

$$E_i = \partial \phi / \partial x_i$$

where u_i are the displacement field components and ϕ is the electric potential.

The generalized Hellinger-Reissner's functional for the domain Ω of boundary $\partial\Omega$ is:

$$\begin{aligned} HR\left(u_i^*, \phi^*, \sigma_{jk}^*, D_l^*\right) &= \int_\Omega \left[-\sigma_{ij}^* \frac{\partial u_i^*}{\partial x_j} + D_i^* \frac{\partial \phi^*}{\partial x_i} \right] d\Omega \\ &+ \int_\Omega \left[\frac{1}{2}\sigma_{ij}^* S_{ijkl}^D \sigma_{kl}^* + \sigma_{ij}^* g_{ijk} D_k^* - \frac{1}{2} D_i^* \beta_{ij}^\sigma D_j^* \right] d\Omega \\ &+ \int_{\partial\Omega_T} \left(T_i^d u_i^*\right) dS + \int_{\partial\Omega_u} \left(\sigma_{ij}^* n_i \left(u_j^* - u_j^d\right)\right) dS \\ &+ \int_{\partial\Omega_D} \left(D^d \phi^*\right) dS + \int_{\partial\Omega_\phi} \left(D_i^* n_i \left(\phi^* - \phi^d\right)\right) dS \end{aligned}$$

where,:

$\partial\Omega_T \; Y \; \partial\Omega_u = \partial\Omega$, $\partial\Omega_D \; Y \; \partial\Omega_\phi = \partial\Omega$

n_i = components of the normal vector on $\partial\Omega$.

S_{ijkl}^D, g_{ijk} and β_{ij}^σ are the components of the mechanical compliance, and of the piezoelectric and dielectric behavior.

T_i^d, u_i^d, D^d, ϕ^d are the given stress vector, displacement, normal electric displacement, electric potential components respectively on $\partial\Omega_T$, $\partial\Omega_u$, $\partial\Omega_D$, $\partial\Omega_\phi$.

It is easy to show that the stationary point of the Hellinger-Reissner's functional is the set of fields $\left(u_i, \phi, \sigma_{jk}, D_l\right)$ such that:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \frac{\partial D_i}{\partial x_i} = 0, \quad \varepsilon_{ij} = S_{ijkl}^D \sigma_{kl} + g_{ijk} D_k$$

$$E_i = -g_{kli}\sigma_{kl} + \beta_{ij}^\sigma D_j, \quad in\ \Omega$$

$$\sigma_{ij} n_j = T_i^d \text{ on } \partial\Omega_T,\ u_i = u_i^d \text{ on } \partial\Omega_u,\ D_i n_i = D^d \text{ on } \partial\Omega_D \text{ and } \phi = \phi^d \text{ on } \partial\Omega_\phi.$$

Theorem: the stationary point $\left(u_i, \phi, \sigma_{jk}, D_l\right)$ *of the Hellinger-Reissner's functional is the solution of the 3D piezoelectric problem.*

3. SIMPLIFICATION OF THE 3D PROBLEM

One way to obtain a simplified model, when we have a generalized Hellinger-Reissner's formulation is to follow the method used by Pagano for the classical laminated plates [2].

Polynomial approximations in the thickness of the stress fields are chosen for each layer.
We can do the same thing for piezoelectric laminate by using polynomial approximations in the thickness of each layer, of the stress fields and the electric displacement fields. We shall present here the simplest of such approximated models. Each layer is considered as a membrane, it means without bending stiffness.
Let us consider a laminated plate with n layers and let us choose a frame such that the layer $c, (c \in \{1,...,n\})$, is given by $x_3 \in [h_c^-, h_c^+]$, (with $h_c^+ = h_{c+1}^-$).

3.1. Generalized "stress" fields

We introduce first the definitions of what we called the generalized fields of the simplified model, mechanical stresses, and electric "stresses" (relative to electric displacement):

-The components, ($\alpha, \beta \in \{1,2\}$), of membrane stress in layer $c, (c \in \{1,...,n\})$,

$$N_{\alpha\beta}^c(x_1, x_2) = \int_{h_c^-}^{h_c^+} \sigma_{\alpha\beta}(x_1, x_2, x_3)\, dx_3$$

-The components, ($\alpha \in \{1,2\}$), of global electric displacement in layer $c, (c \in \{1,...,n\})$,

$$\Delta_\alpha^c(x_1, x_2) = \int_{h_c^-}^{h_c^+} D_\alpha(x_1, x_2, x_3)\, dx_3$$

-The shear stress components, ($\alpha \in \{1,2\}$), at the interface $(c, c+1)$, $(c \in \{0,...,n\})$,

$$\tau_\alpha^{c,c+1}(x_1, x_2) = \sigma_{\alpha 3}(x_1, x_2, h_c^+)$$

Often, $\tau_\alpha^{0,1}(x_1, x_2)$ and $\tau_\alpha^{n,n+1}(x_1, x_2)$ are data.

-The scalar normal dielectric displacement at the interface $(c, c+1)$, $(c \in \{0,...,n\})$

$$D_3^{c,c+1}(x_1, x_2) = D_3(x_1, x_2, h_c^+)$$

The <u>multi-membrane approximation</u> is

$$(x_3 \in [h_c^-, h_c^+]),\ (\alpha, \beta \in \{1,2\}):$$

$$\sigma_{\alpha\beta}(x_1, x_2, x_3) = \frac{N_{\alpha\beta}^c(x_1, x_2)}{e_c}$$

$$D_\alpha(x_1, x_2, x_3) = \frac{\Delta_\alpha^c(x_1, x_2)}{e_c}$$

$$\sigma_{\alpha 3}(x_1, x_2, x_3) = \frac{\tau_\alpha^{c-1,c}(x_1, x_2) + \tau_\alpha^{c,c+1}(x_1, x_2)}{2}$$

$$-\left(\tau_\alpha^{c-1,c}(x_1, x_2) - \tau_\alpha^{c,c+1}(x_1, x_2)\right)\frac{(x_3 - \overline{h}_c)}{e_c}$$

$$D_3(x_1,x_2,x_3)=\frac{D_3^{c-1,c}(x_1,x_2)+D_3^{c,c+1}(x_1,x_2)}{2}$$
$$-\left(D_3^{c-1,c}(x_1,x_2)-D_3^{c,c+1}(x_1,x_2)\right)\frac{(x_3-\overline{h}_c)}{e_c}$$

$\sigma_{33}(x_1,x_2,x_3)$ is given by an integration of the third 3D equilibrium equation.
where $e_c=h_c^+-h_c^-$ is the thickness of the layer c, and $\overline{h}_c=(h_c^++h_c^-)/2$.

3.2. Generalized associate fields

Putting these approximations in the generalized Hellinger-Reissner's functional gives the following associated fields:

The membrane mean displacement components of the layer $c,(c\in\{1,...,n\})$

$$U_\alpha^c(x_1,x_2)=\frac{1}{e_c}\int_{h_c^-}^{h_c^+}u_\alpha(x_1,x_2,x_3)\,dx_3$$

The global bending displacement:

$$W_3(x_1,x_2)=\left(u_3(x_1,x_2,h_n^+)+u_3(x_1,x_2,h_1^-)\right)/2$$

The mean electric potential of the layer
$c,(c\in\{1,...,n\})$

$$\phi^c(x_1,x_2)=\frac{1}{e_c}\int_{h_c^-}^{h_c^+}\phi(x_1,x_2,x_3)\,dx_3$$

The outer surface electric potentials

$$\phi^0(x_1,x_2)=\phi(x_1,x_2,h_1^-)$$
$$\phi^{n+1}(x_1,x_2)=\phi(x_1,x_2,h_n^+)$$

Either $D_3^{0,1}$ ($D_3^{n,n+1}$) or ϕ^0 (ϕ^{n+1}) are data.

3.3. generalized equilibrium equations

The stationary of the generalized Hellinger-Reissner's functional gives first the "generalized equilibrium equations":

$$\frac{\partial N_{\alpha\beta}^c}{\partial x_\beta}-\left(\tau_\alpha^{c-1,c}-\tau_\alpha^{c,c+1}\right)=0,\quad \alpha\in\{1,2\}$$
$$\sum_{c=1}^{n}\frac{e_c}{2}\frac{\partial\left(\tau_\alpha^{c-1,c}+\tau_\alpha^{c,c+1}\right)}{\partial x_\alpha}+T_3^++T_3^-=0$$
$$\frac{\partial\Delta_\alpha^c}{\partial x_\alpha}-\left(D_3^{c-1,c}-D_3^{c,c+1}\right)=0$$

and the boundary conditions:

$$N_{\alpha\beta}^{c} \cdot n_{\alpha} = \int_{h_c^-}^{h_c^+} T_{\alpha}^{d} dz$$

$$\sum_{c=1}^{n} \frac{e^{c}}{2} \cdot \left(\tau_{\alpha}^{c,c+1} + \tau_{\alpha}^{c-1,c}\right) \cdot n_{\alpha} = \int_{h_1^-}^{h_n^+} T_{3}^{d} dz$$

$$\Delta_{\alpha\beta}^{c} \cdot n_{\alpha} = \int_{h_c^-}^{h_{ci}^+} D_{\alpha}^{d} dz$$

$$\phi^{c} = \int_{h_c^-}^{h_c^+} \frac{\phi^{d}}{e^{c}} dz$$

where n_{α} are the components of the normal vector on the edge.

3.4. generalized strains and constitutive equations

The stationary of the generalized Hellinger-Reissner's functional gives a linear constitutive relationship between the "generalized stresses" $N_{\alpha\beta}^{c}, \Delta_{\alpha}^{c}, \tau_{\alpha}^{c,c+1}, D_{3}^{c,c+1}$ and the following "generalized strains":

generalized strains

The in plane tensor components of the membrane strain in the layer $c, (c \in \{1,...,n\})$

$$\varepsilon_{\alpha\beta}^{c} = \left(\frac{\partial U_{\alpha}^{c}}{\partial x_{\beta}} + \frac{\partial U_{\beta}^{c}}{\partial x_{\alpha}} \right) / 2$$

The in plane electrical mean field components in the layer $c, (c \in \{1,...,n\})$

$$E_{\alpha}^{c} = \frac{\partial \phi^{c}}{\partial x_{\alpha}}$$

The interface shear generalized strain at the interface $(c, c+1), (c \in \{1,...,n-1\})$

$$D_{\alpha}^{c,c+1} = \left(U_{\alpha}^{c+1} - U_{\alpha}^{c} + \frac{e_{c+1} + e_{c}}{2} \frac{\partial W}{\partial x_{\alpha}} \right)$$

The global difference of electric potential $\left(\phi^{c+1} - \phi^{c}\right), (c, c+1), (c \in \{0,...,n\})$.

Generalized constitutive relations

$$\varepsilon^{c}_{\alpha\beta} = \frac{S^{c}_{\alpha\beta\gamma\delta}}{e^{c}} \cdot N^{c}_{\gamma\delta} + \tfrac{1}{2}\left(g^{c}_{3\gamma\delta}\right)\left(D_3^{c-1,c} + D_3^{c,c+1}\right)$$

$$d^{c,c+1}_{\alpha\beta} = e^{c}\,\frac{2S^{c}_{\alpha 3}}{3} \cdot \tau_{\alpha}^{c-1,c} + \tfrac{1}{3}\left(e^{c} \cdot 4S^{c}_{\alpha 3} + e^{c+1} \cdot 4S^{c+1}_{\alpha 3}\right.$$

$$+ e^{c+2}\,\frac{S^{c+2}_{\alpha 3}}{3} \cdot \tau_{\alpha}^{c+1,c+2} + \frac{2g^{c}_{\alpha\beta 3} \cdot \Delta^{c}_{\alpha\beta} + 2g^{c+1}_{\alpha\beta 3} \cdot \Delta^{c+1}_{\alpha\beta}}{2}$$

$$\phi^{c}_{,\alpha} = \frac{\beta^{c}_{\alpha\beta}}{e^{c}}\Delta^{c}_{\beta} + \frac{g^{c}_{\alpha\beta 3}}{2}\left(\tau_{\alpha}^{c-1,c} + \tau_{\alpha}^{c,c+1}\right)$$

$$\phi^{c+1} - \phi^{c} = -\frac{\beta^{c}_{33}e^{c}}{6}D_3^{c-1,c} - \frac{\left(\beta^{c}_{33}e^{c} + \beta^{c+1}_{33}e^{c+1}\right)}{3}D_3^{c,c+1}$$

$$-\frac{\beta^{c+1}_{33}e^{c+1}}{6}D_3^{c+1,c+2} + \frac{g^{c}_{3\alpha\beta} \cdot N^{c}_{\alpha\beta} + g^{c+1}_{3\alpha\beta} \cdot N^{c+1}_{\alpha\beta}}{2}$$

We obtain then a simple linear plane model for the linear piezoelectric laminate. However note:

1. the coupling between electric and mechanic aspects
2. the coupling between the different interfaces
   /c-1,c/ - /c,c+1/ - /c+1, c+2/

This set of equations permits the analytical resolution of academic problems.
For example, if we are interested in cylindrical bending of piezoelectric plates, we have only linear differential equations to solve.

4. LAMINATED PIEZOELECTRIC PLATES UNDER CYLINDRICAL BENDING

An interesting situation is the study of the laminated piezoelectric plate bearing force density or electric potential under cylindrical bending, because the problem can be solved with Fourier series. The reference 3D exact solutions exist as for the non-piezoelectric laminates (Pagano [4]). Analytical and numerical results are given in Heyliger [6][7].

4.1. Piezoelectric bi-layer structure with a sinusoidal electric potential applied to the upper face (actuator)

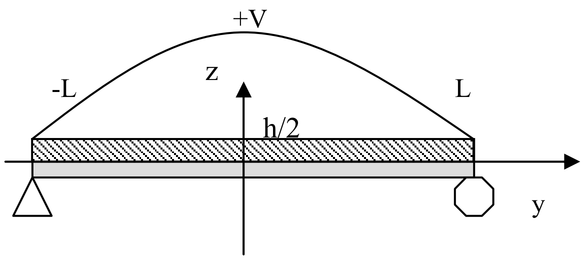

Figure 1: Piezoelectric bi-layer structure with a sinusoidal electric potential applied to the upper face (actuator)

The two layers are piezoelectric with same symmetries, but different materials.
The loading is:

$\phi^d = 0$ on the bottom of the plate

$\phi^d(y) = V = \cos\left(\frac{\pi y}{2L}\right)$ on the top

$\phi^d = 0$ on lateral edges

and no mechanical loading.

The components ($\alpha = 1,2$) of the unknowns are for the two layers (i=1,2):

$$U_\alpha^i(y), W_3(y),$$
$$\phi^i(y), \Delta_\alpha^i(y),$$
$$D_3^{0.1}(y), D_3^{1,2}(y), D_3^{2,3}(y)$$
$$N^i{}_{\alpha\beta}(y), \tau_\alpha^{1,2}$$

(note the presence of the interface shear stresses).

and the boundary conditions gives:

$$N^i{}_{12}(\pm L) = N^i{}_{22}(\pm L) = 0$$
$$\tau_2^{1,2}(\pm L) = 0$$
$$\phi_1(\pm L) = \phi_2(\pm L) = 0$$

Seeing that the applied potential is chosen as a cos function, solutions for the unknowns are sought in the form (taking into account symmetries of the problem):

$$U_\alpha^i(y) = \overline{U}_\alpha^i \sin\left(\frac{\pi y}{2L}\right), \quad W_3(y) = \overline{W}_3 \cos\left(\frac{\pi y}{2L}\right)$$
$$\phi^i(y) = \overline{\phi}^i \cos\left(\frac{\pi y}{2L}\right)$$
$$\Delta_\alpha^i(y) = \overline{\Delta}_\alpha^i \sin\left(\frac{\pi y}{2L}\right), \quad D_3^{i,i+1}(y) = \overline{D}_3^{i,i+1} \cos\left(\frac{\pi y}{2L}\right)$$
$$N^i{}_{\alpha\beta}(y) = \overline{N}^i{}_{\alpha\beta} \cos\left(\frac{\pi y}{2L}\right), \quad \tau_\alpha^{i,i+1} = \overline{\tau}_\alpha^{i,i+1} \sin\left(\frac{\pi y}{2L}\right)$$

Such a solution is found easily and verifies boundary conditions.

This study of a piezoelectric bi-layer shows clearly the mechanical assumptions of the model and its limits. Each layer is considered as a membrane, it means without bending stiffness. However, due to the interface shear stresses, a global bending stiffness for the laminates is effective. But in our 2 layers, equilibrium equations give immediately that $N^{i}{}_{\alpha\beta}(y), \tau_{\alpha}^{1,2}$ are equal to zero, and so, no global bending stiffness. It doesn't make any sense, and therefore the simulation is not relevant. We have to consider at least 3 layers with this model.

4.2. Piezoelectric 2n-layer structure with a sinusoidal electric potential applied to the upper face (actuator)

It represents exactly the same problem as the previous one, with more unknowns, but an analytical solution can also be obtained. Here, $N^{i}{}_{\alpha\beta}(y)$ and $\tau_{\alpha}^{i,i+1}$ are non-zero.

To test the relevance of the model, the previous case (bi-layer, fig.1) was simulated by increasing artificially the number of layers. Actually, it should increase the precision of the results, as does a finest meshing in finite element calculations.
The following figures give for example, the distribution of the interface transverse shear (fig.2), of the interface normal electric displacement (fig.3), of the electric potential (fig.4) in the thickness of a bi-layer, for both the 3D exact solution and the present modelling (with 1,2 or 4 kinematic per physical layer)

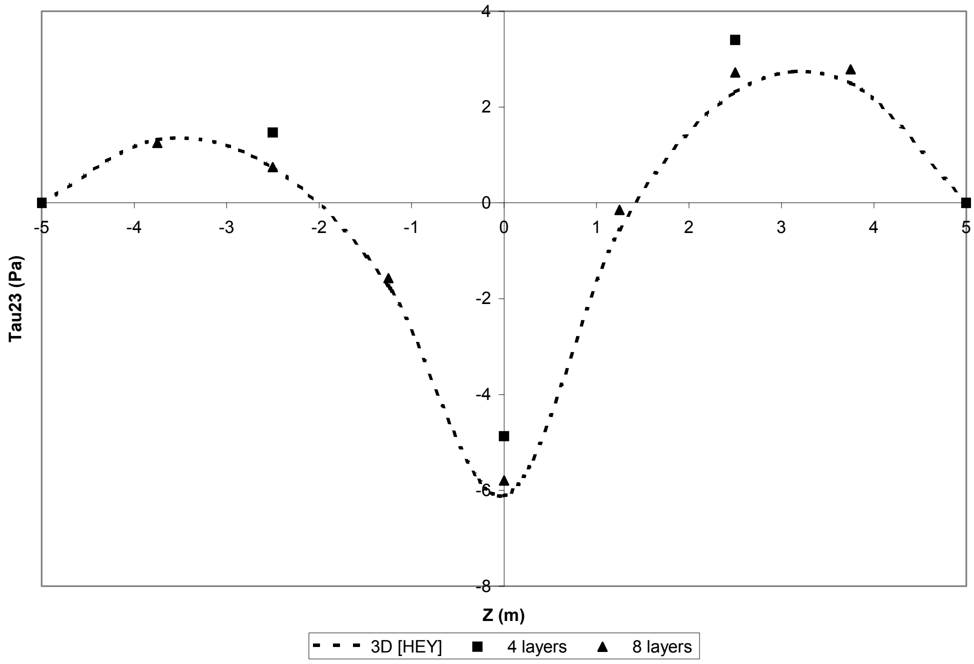

Figure 2: distribution of the interface transverse shear in the thickness of a piezoelectric bi-layer (description with 2 or 4 kinematic per "real" layer)

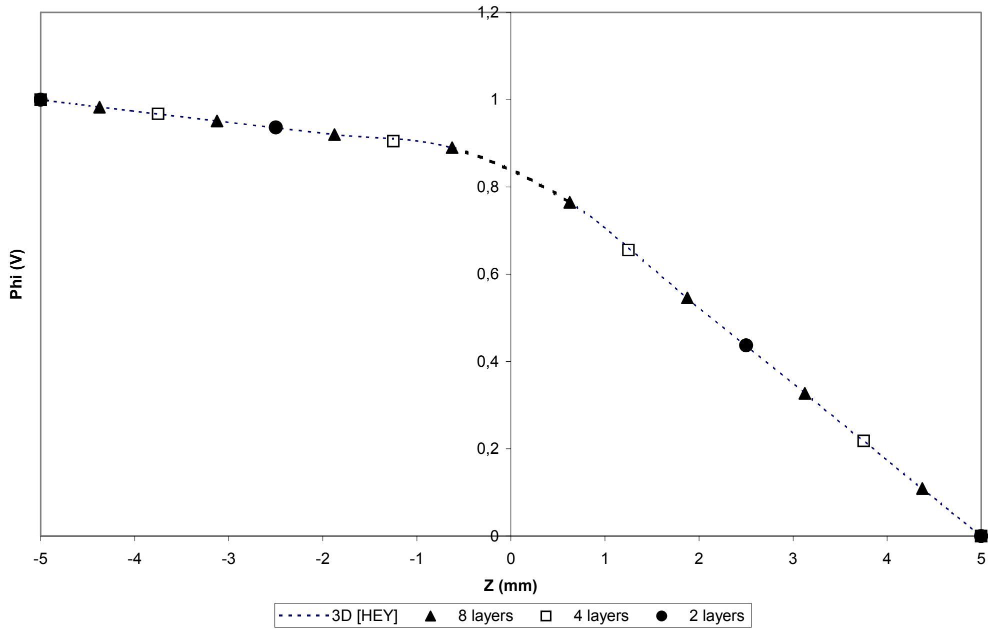

Figure 3: distribution of the electric potential in the thickness of a piezoelectric bi-layer (description with 1, 2 or 4 kinematic per "real" layer)

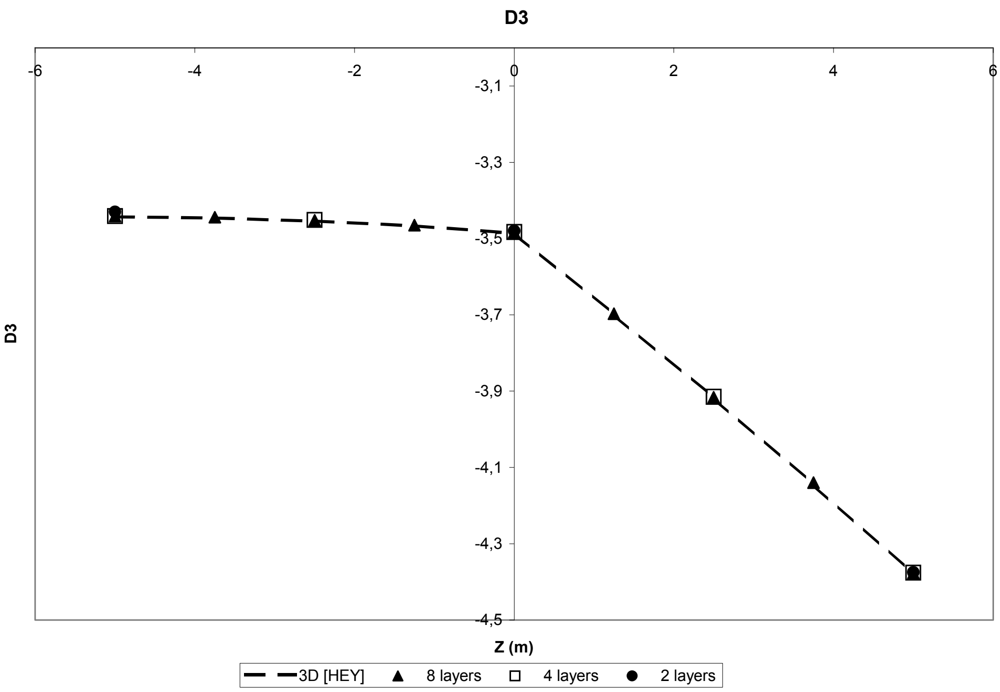

Figure 4: distribution of the interface normal electric displacement in the thickness of a piezoelectric bi-layer (description with 1, 2 or 4 kinematic per "real" layer)

5. CONCLUSIONS

This model of multilayered materials was developed previously to determine interlaminar stresses in multilayered non-piezoelectric composite plates [1,2,3] often responsible of delaminations. This theory is a stress approach, and uses the generalized Hellinger-Reissner's mixed variational function. Some approximations are made about the z variations of the 3D stresses, but no kinetic approximation is made. The aim of the present work is to extend this formulation to piezoelectric multilayered materials. 3 mechanic generalized stress fields and 2 electric generalized fields of induction are introduced in each layer. 2 mechanic generalized stress fields and 1 electric generalized field of induction are introduced at each interface. It constitutes the simplest choice according to the method, each layer is considered as a membrane (without bending stiffness). This theory leads to the determination of generalized fields (in particular, a global deflection), gives generalized bounding conditions, equilibrium equations and constitutive relations. These equations take explicitly into account the coupling between electric and mechanic quantities. Moreover, their relative simplicity can serve to resolve analytically some piezoelectric problems as the piezoelectric 2n-layer structure with a sinusoidal electric potential applied to the upper face (actuator). The validity and the relevance of the model was tested by cutting artificially the 2 layers of the piezoelectric bi-layer, in 2,3 or 4 layers. It was shown that the results of calculations converge very quickly to the exact solutions.

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