A STOCHASTIC APPROACH TO THE FIBRE MICRO-BUCKLING PROBLEM IN LAYERED COMPOSITES

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ABSTRACT

Computational analysis of composites may contain numerous sources of uncertainty: physical properties, direction and alignment of reinforcing fibres, interfacial defects between various constituents. In this paper, a quantification of this uncertainty is attempted using the stochastic field theory. The influence of random fibre waviness on the fibre micro-buckling problem is investigated. Within the framework of the piecewise-homogeneous medium model, it is assumed that the geometry of fibres is described by spatially varying homogeneous stochastic fields with prescribed probability distribution and correlation structure. The critical load/strain variability due to the randomness in fibre geometry is examined and useful conclusions are provided regarding the effect of various random field parameters on this variability.

1. INTRODUCTION

One of the most interesting and inadequately investigated phenomena in mechanics of composites is fracture (instability) in compression when mechanisms of failure, specific to heterogeneous media only, are revealed. A better understanding of these compression failure mechanisms is therefore crucial to the development of improved composite materials. Previous studies [1] have revealed that a possible mechanism of failure initiation is fibre or layer micro-instability (micro-buckling) that may occur in regions where high stress gradients exist, for instance, on the edge of a hole or near free edges. In this study, a stochastic analysis of this internal instability is addressed using the concept of random functions for the modeling of geometric uncertainty.

In mechanics of heterogeneous media, there are two different methods to describe fibre micro-instability: the continuum approach and the piecewise-homogeneous medium model. In the latter, the behavior of each material constituent is described by 3-D equations of solid mechanics. This approach enables the most rigorous investigation of the deformation phenomena that occur in the composite microstructure. However, the complexity of this method makes often the continuum theory more attractive since it involves significant simplifications: the composite is modelled as a homogeneous anisotropic solid with effective stiffness/strength properties. The special mathematical homogenisation theory addressing averaging of the coefficients of equations is very well developed today (see e.g. [2]). Comparisons of the results obtained from the two methods are reported in [3] for layered materials undergoing finite or small deformations and in [4] for non-linear composites undergoing large deformations.

Computational analysis of composites may contain numerous sources of uncertainty: physical properties, direction and alignment of reinforcing fibres, interfacial defects between various constituents. Within the existing deterministic models, it is not possible to predict variation of these parameters. In this work, the quantification of this uncertainty is attempted using the stochastic field theory. The influence of random fibre waviness on the fibre micro-buckling problem is investigated. Within the framework of the piecewise-homogeneous medium model, it is assumed that the geometry of fibres is described by spatially varying homogeneous stochastic fields with prescribed probability distribution and correlation structure. This approach has been applied by the last two authors in [5] for the computation of the response statistics of isotropic shells with multiple random material and geometric properties. In this study, the critical load/strain variability due to the randomness in fibre geometry is examined and useful conclusions are provided regarding the effect of various random field parameters (coefficient of variation, correlation structure) on this variability.

2. MICRO-BUCKLING OF FIBRE COMPOSITES UNDER LARGE DEFORMATIONS

2.1 Exact analytical solution via the piecewise-homogeneous medium model

The section begins with the statement of the stability problem (micro-buckling) for composites undergoing large deformations. Consider a layered composite, where the layers of thickness $2h_r$ and $2h_m$ (Fig. 1) are behaving as incompressible non-linear-elastic isotropic or orthotropic solids. The material is uniaxially compressed in the plane of layers by “dead” loads applied at infinity in such a manner that equal shortening along all layers is achieved and, therefore, the plain strain problem should be considered. Since the analysis in this paper is based on the general approach developed in [6], some equations derived earlier are given below for clarity sake.

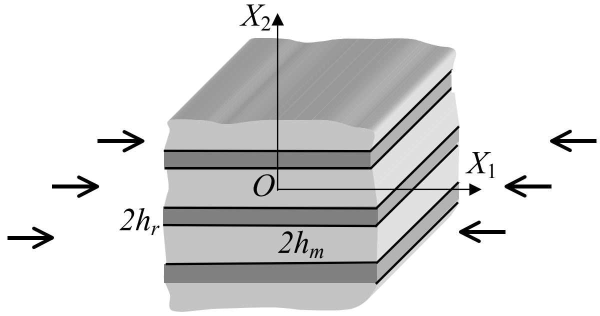

Fig. 1: The coordinate system and applied loads (uniform uniaxial compression)

Within the scope of the most accurate approach (i.e. using the model of piecewise-homogeneous medium) the following eigenvalue problem has to be solved. All values corresponding to the pre-critical state will be marked by the superscript ‘0’ to distinguish them from perturbations of the same values. Indices r and m show that the value is relevant to fibre or matrix, respectively. The axial displacement and strain (in terms of the shortening factor λ_j in the direction of axis OX_j) for the considered type of loading is

$$u_i^0 = (\lambda_i - 1)x_i, \quad \lambda_i = const, \quad \varepsilon_{ij}^0 = (\lambda_i - 1)\delta_{ij}, \quad \text{where } \delta_{ij} \text{ is Kronecker symbol} \qquad (1)$$

The stability equations for the individual layers are

$$\frac{\partial}{\partial x_i} t_{ij}^r = 0, \quad \frac{\partial}{\partial x_i} t_{ij}^m = 0 \qquad (2)$$

The non-symmetrical stress tensor t_{ij} is referred to the unit area of the relevant surface elements in the undeformed state (in the reference configuration). Namely, t_{ij} is a stress component acting in direction of OX_j at the elementary area with normal OX_i. This is the non-symmetrical Piola-Kirchhoff stress tensor or nominal stress tensor using the terminology of Hill [7]. Further we shall consider also the symmetrical stress tensor S_{ij} which reduces to σ_{ij} for the case of small pre-critical deformations. For incompressible solids, stresses are connected with displacements as (p is hydrostatic pressure)

$$t_{ij} = \kappa_{ij\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} + \delta_{ij} q_j p, \quad q_i = \lambda_i^{-1} \text{ with the incompressibility condition } \lambda_1 \lambda_2 \lambda_3 = 1 \qquad (3)$$

Components of tensor $\boldsymbol{\kappa}$ depend on material properties and on loads (i.e. on pre-critical state). The quantity characterising the pre-critical state, i.e. stress S_{ij}^0 or strain ε_{ij}^0, is the parameter with respect to which the eigenvalue problem should be solved. In the most general case

$$\kappa_{ij\alpha\beta} = \lambda_j \lambda_\alpha [\delta_{ij} \delta_{\alpha\beta} A_{\beta i} + (1 - \delta_{ij})(\delta_{i\alpha} \delta_{j\beta} \mu_{ij} + \delta_{i\beta} \delta_{j\alpha} \mu_{ji})] + \delta_{i\beta} \delta_{j\alpha} S_{\beta\beta}^0 \qquad (4)$$

where A_{ij} and μ_{ij} are the quantities which characterise the axial and shear stiffness respectively. Particular expressions of $\kappa_{ij\alpha\beta}$ for numerous kinds of constitutive equations were obtained in [6]. For hyper-elastic solids, if Φ is the strain energy density function (elastic potential)

$$A_{\beta i} = A_{\beta i}(\Phi, \varepsilon_{nl}^0), \quad \mu_{\beta i} = \mu_{\beta i}(\Phi, \varepsilon_{nl}^0) \qquad (5)$$

To complete the problem statement, the boundary conditions should be written for each individual interface

$$t_{22}^r = t_{22}^m, \quad t_{12}^r = t_{12}^m, \quad u_2^r = u_2^m, \quad u_1^r = u_1^m \qquad (6)$$

The detailed problem statement and solutions within the scope of the most accurate approach (as applied to axisymmetrical and non-axisymmetrical modes, biaxial and uniaxial compression) were given previously for the above materials in [6,8].

2.2 Results for hyper-elastic non-linear materials

As an example, let us consider a composite consisting of alternating non-linear elastic isotropic incompressible layers with different properties (Fig. 1). Suppose also that materials of these layers are hyper-elastic (Eq. (5)) and the simplified version of Mooney's potential, namely neo-Hookean potential, may be chosen for their description in the following form

$$\Phi^r = 2C_{10}^r I_1^r(\varepsilon_{ij}^0), \quad \Phi^m = 2C_{10}^m I_1^m(\varepsilon_{ij}^0) \qquad (7)$$

where C_{10} is a material constant, and $I_1(\varepsilon)$ is the first algebraic invariant of Cauchy-Green strain tensor. This potential is also called Treloar's potential, after the author who obtained it from an analysis of model of rubber regarded as a system of long molecular interlinking chains. (For transition to the classical linear theory of elasticity under small deformations, we should put $2C_{10} = G, G = E/3, \nu = 0.5$). Due to the type of applied loads, $\lambda_1^r = \lambda_1^m \equiv \lambda_1$. Since the plane strain state is considered in the pre-critical state, from Eq. (3) we derive $\lambda_3^r = \lambda_3^m = 1$ and $\lambda_2^r = \lambda_2^m = \lambda_1^{-1}$. Then for uniaxial compression the components of the tensor $\boldsymbol{\kappa}$ for this model are expressed, according to [6] and Eqs. (3)-(5), as

$$\kappa_{1111} = 2C_{10}(1 + \lambda_1^{-4}), \quad \kappa_{2222} = 4C_{10}, \quad \kappa_{1212} = 2C_{10}\lambda_1^{-2}, \quad \kappa_{1221} = \kappa_{2112} = 2C_{10}, \quad \kappa_{1122} = 0 \qquad (8)$$

and, therefore, $\eta_2 = \lambda_1^{-2}, \eta_3 = 1$.

Substituting Eq. (8) into the characteristic equations derived in [6,8] for the four considered modes of stability loss, four transcendental equations were deduced for the case of the model of piecewise-homogeneous medium. Then for each of the modes of stability loss we have a different characteristic equation in terms of two variables λ_1 and α_r. For example, after some manipulations, the equations for the 1st (shear) and for the 2nd (extension) modes respectively become

$$-\lambda_1^{-2}(1+\lambda_1^4)^2[1-C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r\lambda_1^{-2} \tanh\alpha_m\lambda_1^{-2} - 4\lambda_1^2[1-C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r \tanh\alpha_m +$$
$$+[2-(1+\lambda_1^4)C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r\lambda_1^{-2} \tanh\alpha_m + [1+\lambda_1^4 - 2C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r \tanh\alpha_m\lambda_1^{-2} +$$
$$+(1-\lambda_1^4)^2 C_{10}^r(C_{10}^m)^{-1}(\tanh\alpha_r \tanh\alpha_r\lambda_1^{-2} + \tanh\alpha_m \tanh\alpha_m\lambda_1^{-2}) = 0 \quad (9)$$

$$-\lambda_1^{-2}(1+\lambda_1^4)^2[1-C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r\lambda_1^{-2} \coth\alpha_m\lambda_1^{-2} - 4\lambda_1^2[1-C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r \coth\alpha_m +$$
$$+[2-(1+\lambda_1^4)C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r\lambda_1^{-2} \coth\alpha_m + [1+\lambda_1^4 - 2C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r \coth\alpha_m\lambda_1^{-2} +$$
$$+(1-\lambda_1^4)^2 C_{10}^r(C_{10}^m)^{-1}(\tanh\alpha_r \tanh\alpha_r\lambda_1^{-2} + \coth\alpha_m \coth\alpha_m\lambda_1^{-2}) = 0 \quad (10)$$

Normalised wavelengths α_r and α_m are related to the layer thickness and to the half-wavelength l of modes of stability loss along the OX_1 axis as

$$\alpha_r = \pi h_r l^{-1} \quad \alpha_m = \pi h_m l^{-1} \quad (11)$$

with α_r representing the fibre waviness.

As the result of solution, four dependences $\lambda_1^{(N)}(\alpha_r)$ are obtained, where $N = 1, 2, 3, 4$ is the number of mode. The critical value for the particular mode $\lambda_{cr}^{(N)}$ can be found as a maximum of the corresponding dependence. The maximal of these four values will be the critical shortening factor of fibre micro-buckling for the considered laminated material determined by the most accurate approach (λ_{cr}).

3. RANDOM FIBRE WAVINESS

A stochastic modelling of imperfect fibre geometry has been attempted in the past by Slaughter and Fleck [9], where the effects of a random misalignment angle (determining the fibre waviness) on the compressive strength of the composite have been studied in the framework of Monte Carlo Simulation (MCS). For the description of the mechanical behavior of the material, the approximate couple stress model was used leading to a differential equation, the solution of which determined the collapse response of the composite for a given distribution of fibre misalignment. In order to perform a parametrical investigation, the authors introduced several different Monte Carlo realizations of imperfect fibre shape in their formulation and each time obtained the corresponding critical micro-buckling stress.

The scope of the present work is to introduce the random fibre waviness in the piecewise-homogeneous medium model presented above. To this end, it is assumed that the geometry of fibres is described by spatially varying homogeneous Gaussian stochastic fields with prescribed correlation structure. This assumption is in accordance with Jelf and Fleck [10] who measured the local fibre misalignment angle and showed that the misalignment angle is roughly Gaussian in nature. Sample functions of the Gaussian fields are generated using the spectral representation method [11].

The spectral representation method is well suited in the context of MCS used for calculating the critical micro-buckling stress/strain variability. The fast Fourier transform (FFT) version of the method is exploited here in order to reduce the computational effort of the simulation. For a 1D-1V stochastic field $g(x)$ and for a specific simulation (i), we have

Series of cosines formula: $g^{(i)}(x) = \sum_{k=0}^{N-1} A_k \cos(\omega_k x + \phi_k^{(i)})$ (12)

FFT version: $g^{(i)}(p\Delta x) = \mathrm{Re}\left[\sum_{k=0}^{M-1} B_k e^{i\left(\frac{kp2\pi}{M}\right)}\right],\ p = 0,1,...,M-1$ (13)

$\phi_0^{(i)}, \phi_1^{(i)}, ..., \phi_{N-1}^{(i)}$ represent the realization for the (*i*) simulation of the independent random phase angles uniformly distributed in the range [0,2π]. The other parameters appearing in Eq. (12) have the following expressions

$$A_k = \sqrt{2S_{gg}(\omega_k)\Delta\omega},\quad \omega_k = k\Delta\omega,\quad \Delta\omega = \frac{\omega_u}{N},\quad k = 0,1,2,...,N-1 \tag{14}$$

N represents the number of intervals in which the wave number axes are subdivided and ω_u is the upper cut-off wave number defining the active region of the spectral density function (power spectrum) S_{gg} of the Gaussian field. B_k is related to A_k and *M* is the number of intervals Δx in the Cartesian axis. The stochastic field generated by (12) and (13) is periodic with period

$$L = \frac{2\pi}{\Delta\omega} = M\Delta x \tag{15}$$

This characteristic is important for our study as it will be shown below.

The two-sided spectral density function S_{gg} used in this study is given by

$$S_{gg}(\omega) = \frac{\sigma^2 a}{\pi}\frac{1}{a^2 + \omega^2},\ a = 1/b \tag{16}$$

where σ is the standard deviation of the stochastic field and *b* denotes a parameter that influences the shape of the spectrum and hence the scale of correlation. It is reminded here that the power spectrum reflects the correlation structure of a random field in the wave number domain. Strictly speaking, the choice of this function is substantiated by available experimental data. The spectral density measurements of Clarke et al. [12] led us to the simple form of Eq. (16) since its values are consistent with those reported in [12].

Through random number generation and using Eq. (13), an ensemble of fibre waviness profiles with different periods *L* is produced from the assumed spectral density (16). These sample functions are generated in accordance with the pre-assigned second-order statistics of the simulated stochastic field (mean and variance). Each Monte Carlo realization of the fibre shape is related to the piecewise-homogeneous medium model via the α_r parameter of Eq. (11). Different periods *L* = n*l* (*l*: the half wavelength of modes of stability loss) lead to different values of α_r parameter and shortening factor λ. The length (period) of sample functions depends on the upper cut-off wave number ω_u defined from Eq. (16) for several values of correlation length parameter *b*.

4. NUMERICAL RESULTS & DISCUSSION

In this section, the influence of random fibre waviness on the micro-buckling strain (shortening factor λ) is investigated in the framework of the piecewise-homogeneous medium model. It is assumed that the imperfect fibre geometry is described by a Gaussian stochastic field with zero mean and unit standard deviation. The corresponding spectral density function is given by Eq. (16). Sample functions with different lengths (periods) are generated with the aid of Eqs. (13), (16) for several values of correlation length parameter b. A typical power spectrum calculated for b = 5.0, ω_u = 3.14 and the corresponding sample function of the stochastic field are depicted in Figs. 2 and 3.

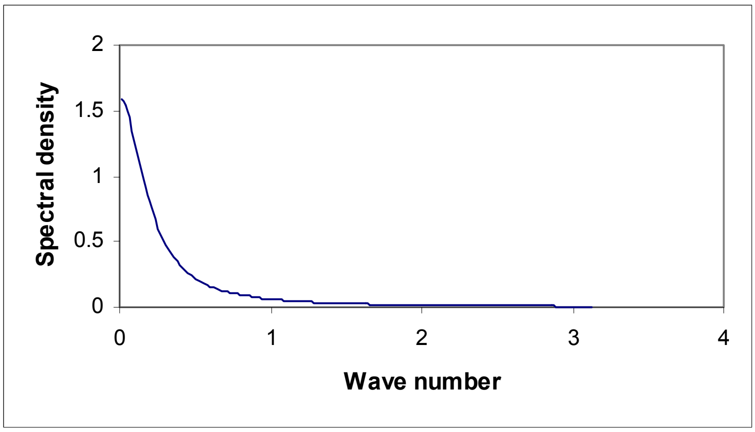

Fig. 2: Typical plot of spectral density function of Eq. (16) for b=5

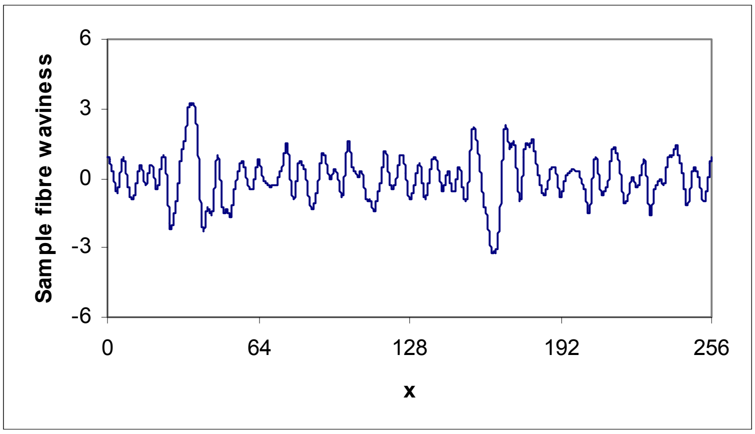

Fig. 3: Sample function of wavy fibre

Typical solutions of the transcendental equations (9), (10) for the first and second modes of stability loss are presented in Table I for different material constants and thickness ratios. These solutions are produced under the assumption of a composite length equal to one wavelength (L = 2l). The results are found to be highly sensitive to the scale of correlation expressed via parameter b and to the upper cut-off wave number ω_u, which finally defines the length L of sample functions.

Table I: Typical values of λ_{cr} and corresponding a_r.

C_{10}^r/C_{10}^m	h_r/h_m	λ_{cr}	α_r	C_{10}^r/C_{10}^m	h_r/h_m	λ_{cr}	α_r
10	0.175	0.8443	0.172	20	0.05	0.8946	0.274
16	0.150	0.8858	0.165	60	0.05	0.9483	0.210
25	0.125	0.9137	0.181	30	0.03	0.9189	0.250
48	0.100	0.9436	0.154	98	0.02	0.9623	0.184

It would be interesting to investigate the effect on the composite response of a random fluctuation of tensor $\boldsymbol{\kappa}$, Eq. (4), representing the mechanical properties of the material. The presence of interfacial defects between various constituents may also influence significantly the micro-buckling behavior of the composite. The inclusion of all the aforementioned parameters in a finite element model will permit the examination of their importance in a more "engineering" way and constitutes a future scope for the authors.

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