AN AUTOMATED COMBINED QUASI-STATIC/PSEUDO-DYNAMIC PROCEDURE FOR CAPTURING MODE-SWITCHING IN POSTBUCKLING STRUCTURES

Brian G. Falzon[1] and Marco Cerini[1]

[1] Department of Aeronautics, Imperial College London, South Kensington Campus, SW7 2AZ London, UK

ABSTRACT

This paper presents a finite element procedure for modelling the behaviour of postbuckling structures undergoing mode-switching. The arc-length method has been shown to be able to capture mode-switching under certain circumstances but it is by no means robust. A more effective strategy is to use a combined static-dynamic analysis for modelling this behaviour. In most studies that are currently reported in the literature, the switching from one solution procedure to the other is performed interactively using restarting schemes available in most finite element packages. In this paper an automated static-dynamic procedure is presented. The static part of the solution process uses the arc-length constraint. A modified explicit dynamic routine, which is more computationally efficient than standard implicit and explicit schemes, is used for the transient phase. Bracketing procedures for determining the location of critical points and eigenmode injection to switch to a secondary equilibrium path were also incorporated in the algorithm. This negated the need of introducing imperfections in the panel to reduce bifurcation points to limit points.

1. INTRODUCTION

Numerous experimental investigations on stiffened composite panels loaded in uniaxial compression have shown their load carrying capability beyond the initial skin-buckling load [1-3]. By allowing certain structural components to buckle between the design limit and ultimate loads, and in some cases even below the design limit load, significantly lighter structures can be achieved. Buckling in stiffened composite aerostructures gives rise to a dramatic increase in the interlaminar stresses at the skin-stiffener interfaces which are only resisted by the relatively weak through-thickness strength of the adhesive or resin. Damage initiation and progression in these vulnerable regions is still difficult to predict and this has resulted in conservative composite designs in aerostructures.

The postbuckling behaviour of stiffened composite structures is further complicated by the observed phenomenon of mode-switching where a postbuckled panel will dynamically snap to a higher mode-shape. Primary skin-buckling is characterised by a stable bifurcation point which defines the intersection of the primary and a secondary stable equilibrium path. Various studies [4,5] have shown that for secondary instabilities, the bifurcation point is the intersection of the current stable equilibrium path with at least one secondary unstable equilibrium path, which eventually leads to a stable post-critical branch. This concept is schematically shown in Fig. 1, where q_m and q_n are generalised coordinates representing a deformation mode.

The arc-length method has been shown to be able to capture this secondary instability (commonly referred to as mode-switching) under certain circumstances but it is by no means robust [7]. A more effective strategy for modelling this behaviour is to use a combined static-dynamic analysis and this was demonstrated by Riks *et al.* [8]. In this study, as in most that are currently reported in the literature, the switching from one solution procedure to the other is performed interactively using restarting schemes available in most finite element packages.

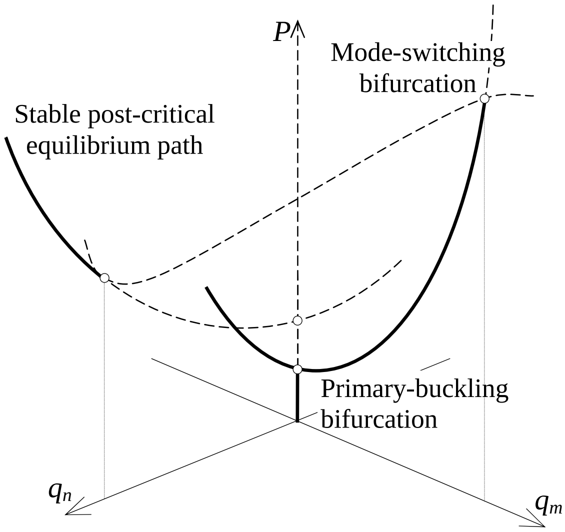
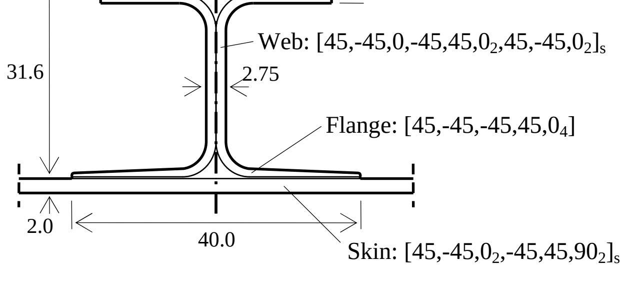


Fig. 1. Equilibrium path showing primary and secondary instabilities (solid lines represent stable states, and broken lines unstable states).

This paper presents an automated static-dynamic procedure for modelling this phenomenon. The static part of the solution process uses the arc-length constraint. A modified explicit dynamic routine, which is more computationally efficient than standard implicit and explicit schemes, is used for the transient phase. Bracketing procedures for determining the location of critical points along an equilibrium path and eigenmode injection to switch to a secondary path were also incorporated in the algorithm, removing the need of using initial imperfections in the panel to reduce bifurcation points to limit points. This is usually done to allow arc-length routines to proceed past these critical regions but assumes that if a mode-jump occurs, the pre- and post-jump stable equilibrium paths are statically connected by an unstable equilibrium path. As pointed out by Riks [8], this is not necessarily the case and hence path-following procedures are liable to fail.

2. EXPERIMENTAL EVIDENCE

A carbon-fibre reinforced plastic (CFRP) panel, manufactured using T300/914C unidirectional prepreg, was tested to failure under uniaxial compression. The panel was 790 mm long and 604 mm wide, and was stiffened by four evenly spaced secondary-bonded stiffeners. The stiffeners cross-section is shown in Fig. 2, together with the lay-up and material properties.



Property	Value
Longitudinal Young's modulus, E_{11}	127.5 GPa
Transverse Young's modulus, E_{22}	9.0 GPa
In-plane shear modulus, G_{12}	4.9 GPa
Poisson's ratio, ν_{12}	0.28
Nominal ply thickness	0.125 mm

Fig. 2. Stiffener cross-section (dimensions in mm) and material properties of unidirectional T300/914C lamina.

The various experimental stages of the panel test are shown in Fig. 3, where the shadow-Moiré technique was employed to visualise out-of-plane displacements. The panel's skin buckled at approximately 119 kN (corresponding to an end-displacement, Δ, of 0.95 mm). At 244 kN (Δ = 1.92 mm) a mode-switch from five half-waves to six half-waves in the skin-bays, was observed. It was however noted that the overall stiffness was hardly affected. At a loading of 473 kN (Δ = 3.96 mm) the left bay switched configuration to seven half-waves, and this was followed by an identical mode-switch in the right bay at a loading of 486 kN (Δ = 4.11 mm). Audible cracks were heard at loading over 300 kN. The panel failed catastrophically at a loading of 525 kN (Δ = 4.60 mm).

Back-to-back strain gauges were located in the middle of each bay and a strain gauge pair was also positioned on the top of the stiffener cap and the stiffener flange to measure overall bending. It was observed that a certain amount of overall bending bowed the panel towards the stringer side. At final failure, however, the panel was bent in the opposite direction. This observation supports the thesis that failure occurred due to a premature local collapse, rather than overall structural buckling.

This and other experimental tests [1-3,6] have shown that damage initiates at either a buckling node or an anti-node line. Mode-switching results in sudden shifts in these critical locations and may even release enough energy to cause damage. If mode switching occurs frequently, delamination may arise as a consequence of fatigue. Oscillations during the transient phase are associated with large stress variations, with possible sign reversal. Hence analytical tools used to predict structural response must be able to capture this phenomenon.

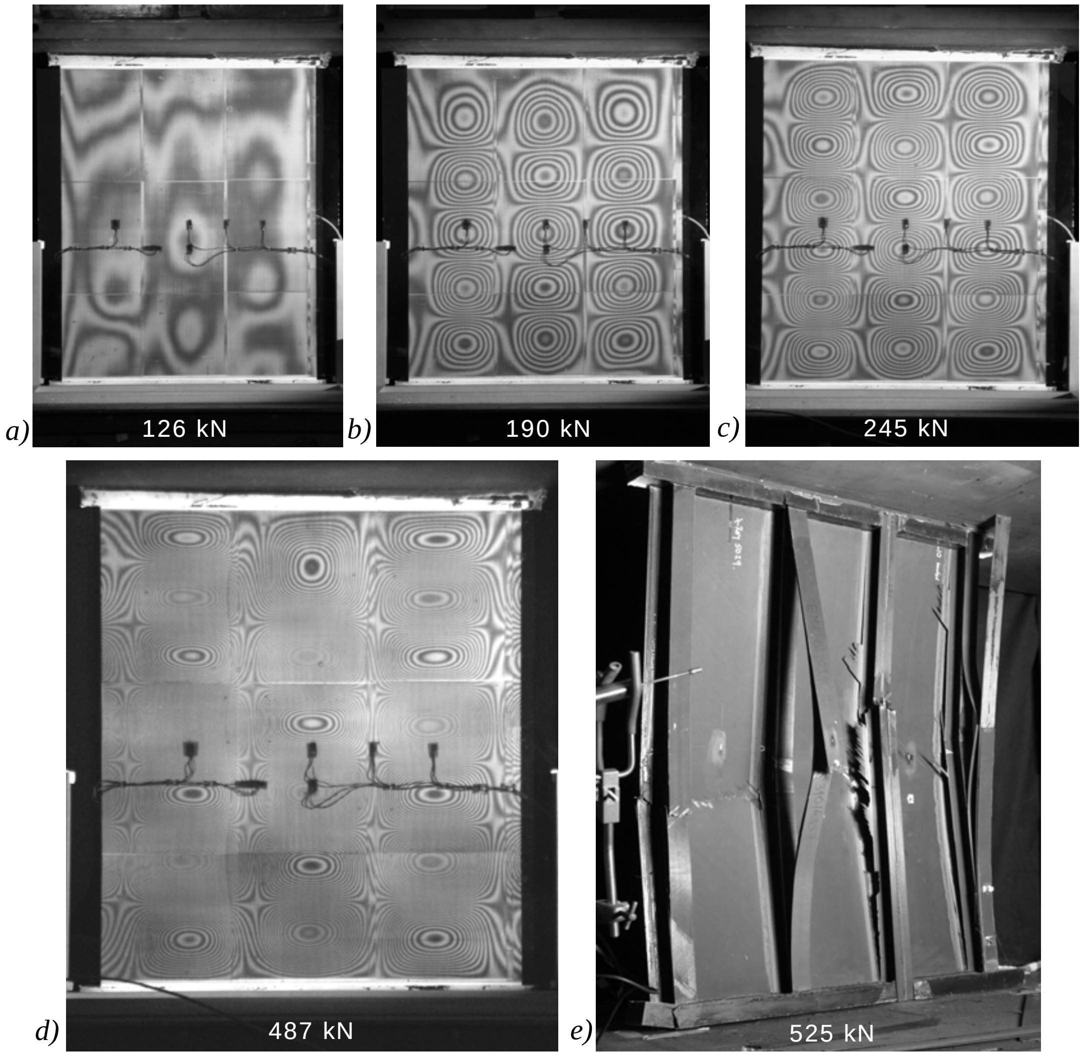


Fig. 3. Moiré fringe patterns of an I-stiffened postbuckling panel.

3. NUMERICAL STRATEGY

Given the static-dynamic nature of problems exhibiting mode-jumping under quasi-static loading, a combined procedure has been developed, which employs the arc-length method in quasi-static phases and a modified explicit dynamic method during transient phases.

Arc-length method

The arc-length method is a well established solution procedure for quasi-static problems that was first introduced by Wempner [13] and Riks [14]. It is based on coupling the system of non-linear equilibrium equations:

$$\mathbf{g}(\mathbf{u},\lambda) := \mathbf{f}(\mathbf{u}) - \lambda\bar{\mathbf{r}} = \mathbf{0} \qquad (1)$$

with an incremental constraint which is a function of both the incremental displacements and the load increment. In Eq. (1), $\mathbf{g}$ is the vector of residual forces, $\mathbf{f}$ the vector of internal forces, function of displacements $\mathbf{u}$, $\bar{\mathbf{r}}$ is a fixed external load vector and λ a load multiplier. This method overcomes a major limitation of the traditional Newton-Raphson scheme in the vicinity of limit points.

The arc-length method adopts, as predictor, the tangential solution from an equilibrium state at a set arc-length distance, Δl. Various corrector techniques can be employed. Riks [14] suggested constraining the corrector iterations on a hyperplane orthogonal to the predictor, which results in a linear constraint equation. Crisfield [9] proposed a quadratic constraint where corrector solutions lie on a hypersphere centred in the last converged solution and with radius Δl (Fig. 4). A third technique, usually referred to as the normal-flow algorithm [15], does not constrain correctors onto a known surface, but requires that the corrector direction be orthogonal to the Davidenko flow, defined as a set of streamlines with constant $\mathbf{g}$. Since this is the direction of steepest descent of $\mathbf{g}$, it should rapidly converge onto the equilibrium curve $\mathbf{g} = \mathbf{0}$.

In previous work [7] the authors investigated the use of the arc-length method when applied to mode-jumping problems. If no special techniques are applied to detect and overcome bifurcation points, a suitable initial imperfection has to be added to the perfect geometry, to transform bifurcations into limit points. It was shown that the arc-length method, as traditionally formulated, could be further improved to deal more effectively with these limit points but that this method was still susceptible to convergence problems.

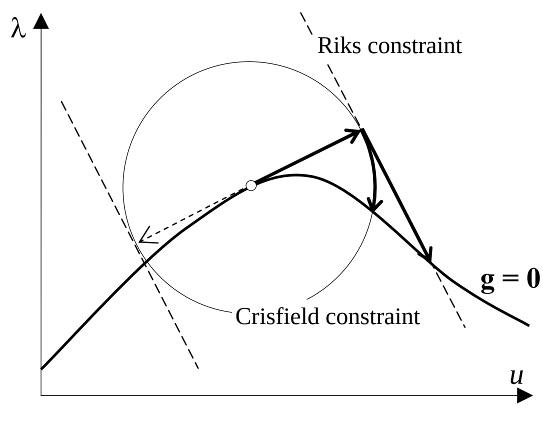


Fig. 4. Arc-length constraints for corrector iterations.

Modified explicit dynamic method

A dynamic approach seems more appropriate to treat mode-switching, a transient dynamic event [16]. Within the available dynamic techniques, explicit methods are very robust but not very efficient in terms of required CPU time.

In modelling postbuckling stiffened composite structures undergoing mode-switching, we are often only interested in locating a stable postcritical equilibrium path without the need of accurately representing the behaviour during the transient phase. A modified explicit procedure was proposed [10] with the aim of accelerating the transient phase that leads to a postcritical stable state. It is known that, given the following linear system of dynamic equilibrium equations:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{r}, \qquad (2)$$

an explicit method is only conditionally stable. The Central Difference Method, for example, is stable provided the size of the time increment δt is lower than or equal to $2/\omega_{\max}$, where $\omega_{\max}$ is the largest natural frequency. To establish static equilibrium, in general, at least one cycle of the fundamental frequency, $\omega_{\min}$, must be completed so that the number of steps during the dynamic analysis is of the order of $\omega_{\max}/\omega_{\min}$. This results in a very high number of time steps as these two frequencies are often orders of magnitude apart.

If the mass matrix **M** is replaced by the stiffness matrix **K**, the ratio $\omega_{\max}/\omega_{\min}$ is drastically reduced to unity. According to these considerations, the modified explicit method for solving the non-linear system:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{f}(\mathbf{u}) = \mathbf{r} \qquad (3)$$

consists of replacing the mass matrix **M** with the tangent stiffness matrix $\mathbf{K}_t := \partial\mathbf{f}(\mathbf{u})/\partial\mathbf{u}$. Damping can then be idealised as proportional (or Rayleigh) as it is often assumed in engineering analyses. It is noted, however, that it is not necessary to update **M** with $\mathbf{K}_t$ at every time step, provided a time step smaller than unity is employed. The efficiency of the method will be a function of the frequency of 'mass-matrix' updates and the critical size of the time increments.

The efficiency of the modified explicit method can be further improved if it is acknowledged that inertial forces do not contribute to the change in mode-shape. Indeed it was found that their inclusion caused gentle oscillations before final convergence. Therefore the use of the following first-order system:

$$\bar{\mathbf{K}}_t\dot{\mathbf{u}} + \mathbf{f}(\mathbf{u}) = \mathbf{r} \qquad (4)$$

lead to faster convergence. In Eq. (4) $\bar{\mathbf{K}}_t$ is the tangent stiffness matrix estimated at a previous increment.

Analysis of stability of the modified explicit method

If the 'damping' matrix is updated at every time step, the condition for the stability of the algorithm for the solution of non-linear system (4) is also such that the time step δt be smaller than or equal to unity.

Let us assume that configuration $\mathbf{u}_0$ is in equilibrium with the external-force vector $\mathbf{r}_0$, and that at the instant $t = t_0$ an additional external force $\Delta\mathbf{r}$ is applied. The velocity vector $\dot{\mathbf{u}}_i$ at time t_i can be directly determined from equation (4) as:

$$\dot{\mathbf{u}}_i = \mathbf{K}_{i-1}^{-1}\left[\mathbf{r}_0 + \Delta\mathbf{r} - \mathbf{f}(\mathbf{u}_i)\right], \tag{5}$$

where the tangent stiffness matrix of the previous increment:

$$\mathbf{K}_{i-1} := \left.\frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}}\right|_{\mathbf{u}_{i-1}} \tag{6}$$

is used (instead of the current stiffness $\mathbf{K}_i$). By assuming a constant velocity within each time step and using the same time increment δt in all steps, the incremental displacement vector will be:

$$\delta\mathbf{u}_i = \dot{\mathbf{u}}_i\ \delta t = \mathbf{K}_{i-1}^{-1}\left[\mathbf{r}_0 + \Delta\mathbf{r} - \mathbf{f}(\mathbf{u}_i)\right]\delta t\,. \tag{7}$$

If displacement increments are small enough, it is possible to approximate $\mathbf{f}(\mathbf{u}_i)$ as follows:

$$\mathbf{f}(\mathbf{u}_i) = \mathbf{f}(\mathbf{u}_0) + \Delta\mathbf{f}_{i-1} + \delta\mathbf{f}_{i-1} \approx \mathbf{f}(\mathbf{u}_0) + \Delta\mathbf{f}_{i-1} + \mathbf{K}_{i-1}\delta\mathbf{u}_{i-1}, \tag{8}$$

where $\Delta\mathbf{f}_p := \mathbf{f}(\mathbf{u}_p) - \mathbf{f}(\mathbf{u}_0)$. Eq. (7) now becomes:

$$\delta\mathbf{u}_i \approx -\delta t\ \delta\mathbf{u}_{i-1} + \delta t\ \mathbf{K}_{i-1}^{-1}\left[\Delta\mathbf{r} - \Delta\mathbf{f}_{i-1}\right]. \tag{9}$$

By recursively applying relation (9), we can express the displacement increment of step i as a function of the initial displacement increment $\delta\mathbf{u}_0$ and of the load $\Delta\mathbf{r}$ applied at $t = t_0$:

$$\delta\mathbf{u}_i \approx (-\delta t)^i\,\delta\mathbf{u}_0 - (-\delta t)^i\,\mathbf{K}_0^{-1}\Delta\mathbf{r} - \sum_{p=1}^{i-1}(-\delta t)^{i-p}\mathbf{K}_p^{-1}\left[\Delta\mathbf{r} - \Delta\mathbf{f}_p\right]. \tag{10}$$

From Eq. (10) it is clear that, in order for the above method to be numerically stable, the time increment δt must be limited to one: for $\delta t > 1$ this algorithm diverges. The critical time step is significantly larger than that for conventional explicit dynamic analysis. In practice, the size of the time-step is dependent on accuracy considerations in converging to the correct post-critical solution.

Combined static-dynamic method

In the combined procedure the arc-length method is used during quasi-static phases, and the modified explicit method is used during transient phases. The dynamic procedure is triggered when a critical point (limit or bifurcation point) is encountered. We are therefore interested, firstly, in locating a critical point within a set tolerance. The tangent stiffness matrix is positive definite before a critical point, singular at the critical point and indefinite beyond this point. The presence of a critical point can be detected during the factorisation of the tangent stiffness matrix. When all pivots are positive, the tangent stiffness is positive definite and therefore the system lies in a stable equilibrium state. If there is at least one negative pivot, then the tangent stiffness has changed to indefinite, indicating that a critical point has been traversed. A bracketing procedure, based for example on a bisection process, is needed to locate the mode-switching point within the desired arc-length distance.

In order to switch to the pseudo-dynamic procedure, the following information is needed: the geometric configuration just before the bifurcation point, a small load increment which would lead to an unstable state on the current path, and a small perturbation in order to divert from the current equilibrium path. The perturbation could be either a suitable initial velocity vector or a displacement increment. In the present work a small displacement increment is injected, which has the shape of the critical eigenmode corresponding to the smallest eigenvalue (this will still be positive, though very close to zero). The first eigenmode can be obtained using a traditional eigenvalue analysis, although a more efficient technique has been recently proposed by Fujii and Noguchi [11], which exploits the properties of the triangular and diagonal matrices obtained from an $\boldsymbol{LDL}^T$ decomposition of the tangent stiffness, without requiring an eigenvalue analysis.

Referring to Eq. (10) it is noted that the dynamic procedure described above can yield very large displacement increments in the first few iterations, since in the neighbourhood of critical points the stiffness matrix is almost singular. In these cases it is convenient to bound the norm of the incremental displacement to a maximum value, for example of the same order of magnitude of the arc-length increments. This dynamic procedure proved very robust and able to recover a stable postcritical equilibrium state. From this state, the non-linear analysis can continue with the arc-length method.

4. EXAMPLE 1: BEAM ON NON-LINEAR ELASTIC FOUNDATION

The present example demonstrates the application of the combined procedure to a beam on a non-linear elastic foundation, shown in Fig. 5, and investigates its efficiency, compared to other solution methods. The beam was modelled using two-node truss elements connected with linear rotational springs and supported on their nodes by extensional springs, whose restoring force is proportional to the cube of their extension. The cubic-spring type of support simulates the behaviour of a simply-supported plate under uniaxial compression. Indeed this system can undergo several mode-jumps with increasing load, as shown in Fig. 6. Results of CPU time as a function of mesh density are shown in Fig. 7 for different solution techniques.

5. EXAMPLE 2: STEIN'S PLATE

A seminal experimental study by Stein [10] investigated the behaviour of a rectangular aluminium plate, subdivided into eleven skin bays and loaded in uniaxial compression. These results have been used by numerous investigators as a benchmark for validating analytical/numerical methods. A skin-bay was modelled using the present algorithm. Material non-linearity was not accounted for and the unloaded edges were fixed from moving in-plane. These boundary conditions differed from those in the actual test where the central skin-bay, in which attention was focused, had unloaded edges which remained straight but allowed to move in-plane.

In Fig. 8 the load vs end-displacement curve obtained with the combined procedure is shown. The same curve is compared with a solution obtained using the arc-length method with an initial imperfection given by the combination of the first two eigenmodes. It is noted that mode-jumps occur at different load levels for the two analyses. Moreover, the combined procedure, which makes use of a perfect initial geometry, predicts a buckling mode with three half-waves, rather than four as given by the arc-length analysis.

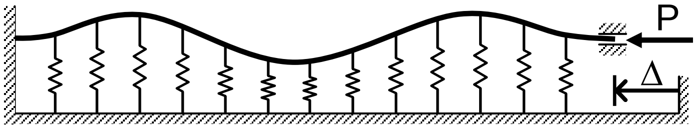


Fig. 5. Beam on a non-linear elastic foundation.

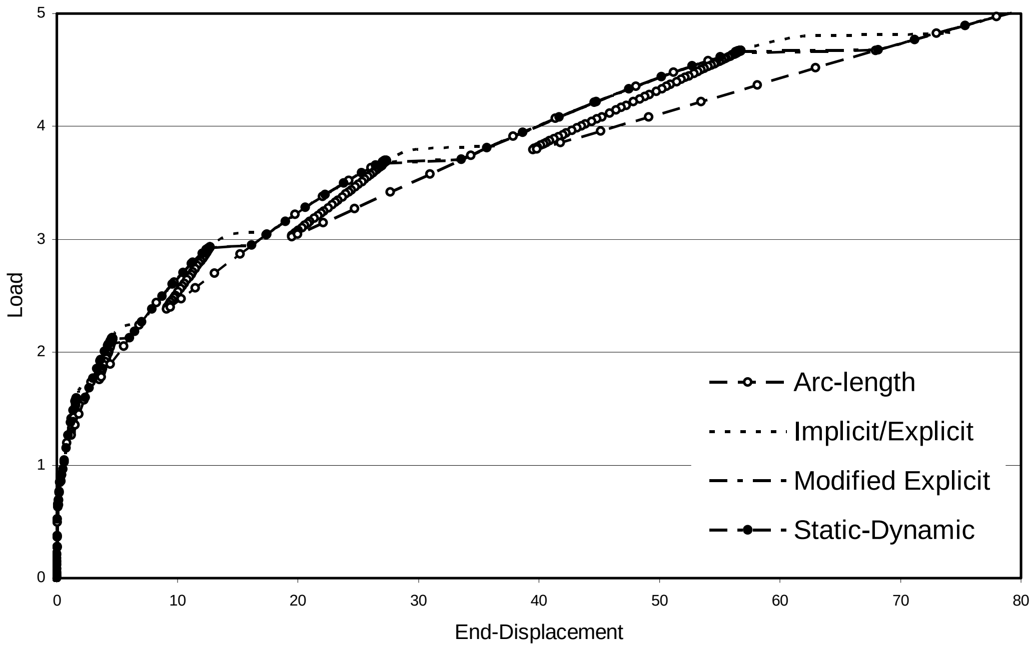


Fig. 6. Load vs end-displacement for beam on non-linear elastic foundation modelled with 80 elements.

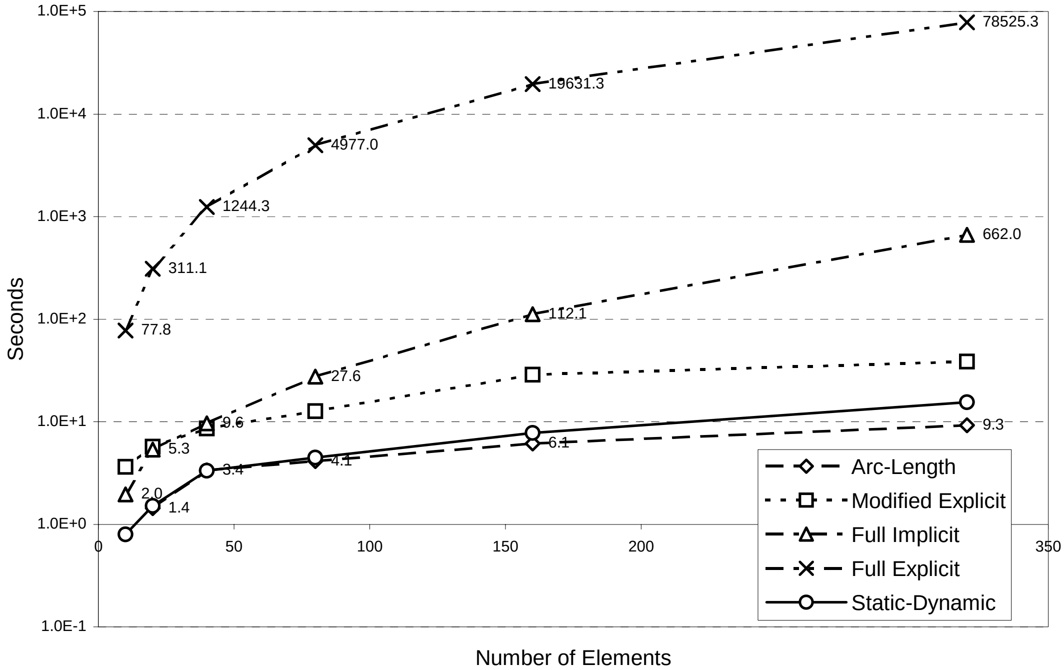


Fig. 7. Comparison of CPU time required by various solution methods, for a beam on elastic foundation.

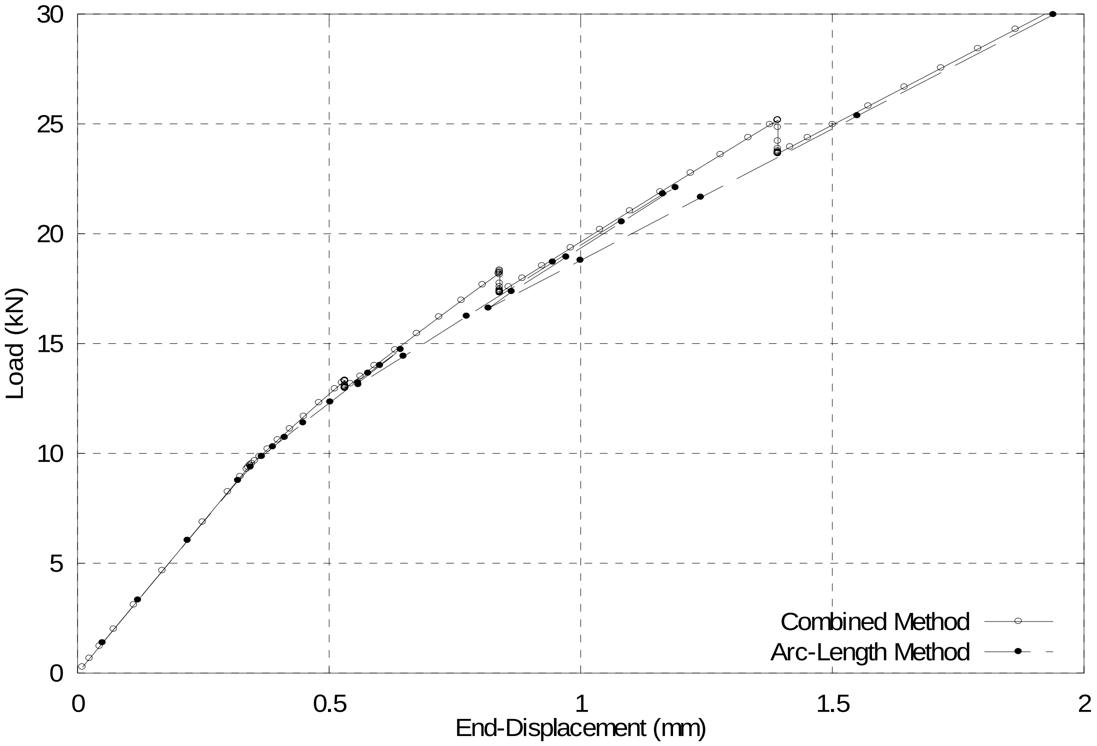


Fig. 8. Load vs end-displacement curve for an aluminium skin-bay.

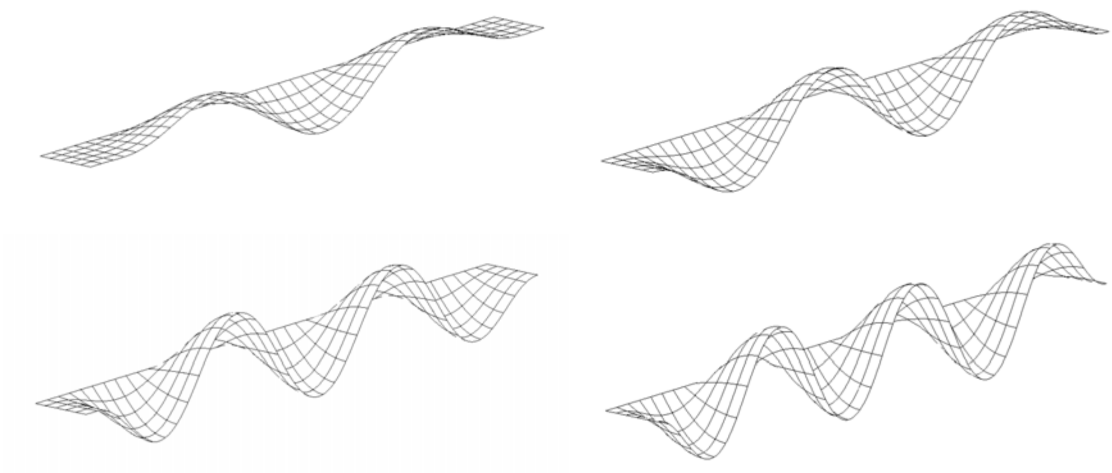


Fig. 9. Deformation of aluminium skin-bay with increasing load, showing mode-switches.

Fig. 9 shows the buckling mode (3 half-waves) and the first three mode-switches (4 – 5 – 6 half-waves) obtained in the present analysis. Mode-switching is highly sensitive to boundary conditions and the first three mode-shapes reported in the experiment were (5 – 6 – 7 half-waves). The panel underwent plastic deformation beyond six half-waves.

6. CONCLUDING REMARKS

A robust and efficient algorithm for capturing the postbuckling behaviour of structures undergoing secondary instabilities has been presented. This method requires no user-intervention, such as restart schemes, to capture this phenomenon. By using a bracketing procedure to locate critical points and followed by eigenmode injection to 'point' the displacement increment in the right direction, no initial imperfections need to be introduced in the initial geometry.

The example of the beam on a non-linear elastic foundation showed that this method was able to capture numerous mode-switches and was computationally more efficient than the full dynamic routines. While the arc-length routine was shown to be the most computationally efficient, convergence difficulties were encountered for one of the test-cases. The second example of the aluminium skin-bay, loaded in uniaxial compression, shows that the method had no difficulty in capturing mode-switching for plated structures and work is currently underway for the modelling of full stiffened composite panels.

ACKNOWLEDGEMENTS

This research forms part of an EPSRC research contract (GR/R32147/01).

References

1. Falzon, B.G., Stevens, K.A. and Davies, G.A.O. (2000): “Postbuckling behaviour of a blade-stiffened composite panel loaded in uniaxial compression”, *Composites Part A: applied science and manufacturing*, 31, pp. 459-468.
2. Falzon, B.G. (2001): “The behaviour of damage tolerant hat-stiffened composite panels loaded in uniaxial compression”, *Composites Part A: applied science and manufacturing*, 32, pp. 1255-1262.
3. Falzon, B.G. and Steven, G.P. (1997): “Buckling mode transition in hat-stiffened composite panels loaded in uniaxial compression”, *Composite Structures*, 37, pp. 253-267.
4. Stein, M. (1959): “The phenomenon of change in buckle pattern in elastic structures”, Technical Report R-39, NASA.
5. Everall, P. R., Hunt, G. W. (2000): “Mode jumping in the buckling of struts and plates: a comparative study”, *International Journal of Non-Linear Mechanics*, 35, pp. 1067-1079.
6. Stevens, K.A. ,Ricci, R., Davies, G.A.O. (2001): “Buckling and postbuckling of composite structures”, *Composites*, 26(3), pp. 189-199.
7. Cerini, M., Falzon, B.G. (2003): “The reliability of the arc-length method in the analysis of mode-jumping problems”, *44th AIAA/ASME/ASCE/AHS Structures, Structural Dynamics and Materials Conference*, Paper AIAA 2003-1621, 7-10 April, Norfolk, Virginia, USA.
8. Riks, E., Rankin, C.C., Brogan, F.A. (1996): “On the solution of mode jumping phenomenon in thin-walled structures”, *Computational Methods in Applied Mechanics and Engineering*, 136, pp. 59-92.
9. Crisfield, M.A. (1981): “A fast incremental/iterative solution of snapping and buckling problems”, *Computers and Structures*, 13, pp. 55-62.
10. Falzon, B.G., Hitchings, D. (2003): “Capturing mode-switching in postbuckling composite panels using a modified explicit procedure”,*Composite Structures*, 60, pp. 447-453.
11. Fujii, F., Noguchi, H. (2002): “The buckling mode extracted from the LDL^T-decomposed larger-order stiffness matrix”, *Communications in Numerical Methods in Engineering*, 18, pp. 459-467.
12. Stein, M. (1959): “Loads and deformations of buckled rectangular plates”, NASA Technical Report R-40.
13. Wempner, G. A. (1971): “Discrete approximation related to nonlinear theories of solids”, *International Journal of Solids and Structures*, 7, pp. 1581-1599.
14. Riks, E. (1979): “An incremental approach to the solution of snapping and buckling problems”, *International Journal of Solids and Structures*, 15, pp. 529-551.
15. Fried, I. (1984): “Orthogonal trajectory accession to the nonlinear equilibrium curve”, *Computer Methods in Applied Mechanics and Engineering*, 47, pp. 283-297.
16. Caputo, F., Esposito, R. Perugini, P., Santoro, D. (2002): “Numerical-experimental investigation on post-buckled stiffened composite panels”, *Composite Structures*, 55, pp.347-357.